Quaternions and Reflections in $\mathbb{R}^4$

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Abstract

Let $\mathcal{H}$ be the real algebra of quaternions, and let S^3 be the set of unit quaternions. For $a, b \in S^3$, define $B_{a,b}(x) = axb$ for $x \in \mathcal{H}$. We show that $B_{a,b}$ is a product of an even number of reflections. Let $O(4)$ be the orthogonal group, and let $PSp(2)$ be a projective symplectic group. The results in this paper extend [2] wherein a group homomorphism $\xi : O(4) \to PSp(2)$ is defined from the reflections of S^3. In this paper, we evaluate the differential of ξ.

1 Introduction

A quaternion has the form $q = q_1 + \boldsymbol{i}q_2 + \boldsymbol{j}q_3 + \boldsymbol{k}q_4$ where q_1, q_2, q_3 and $q_4 \in \mathbb{R}$, and $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. Let $\mathcal{H}$ be the set of all quaternions and let $q \in \mathcal{H}$ be given. Notice that $\mathcal{H}$ is isomorphic to $\mathbb{R}^4$ as real vector spaces, that is, we look at q as $\widehat{q} = [q_1,\ q_2,\ q_3\ ,q_4]^T \in \mathbb{R}^4$. The *conjugate* of q is $\overline{q} = q_1 - \boldsymbol{i}q_2 - \boldsymbol{j}q_3 - \boldsymbol{k}q_4$. The *norm* of q is $||q|| = \sqrt{q\overline{q}} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$. If $q \neq 0$, the *inverse* of q is $q^{-1} = \frac{\overline{q}}{||q||^2}$. We call q a *pure quaternion* if $q_1 = 0$; q is called a *unit quaternion* if $||q|| = 1$. Let p be a pure unit

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quaternion, then one checks that $p^2 = -1$, so that $p^{2n} = (-1)^n$ and $p^{2n+1} = (-1)^n p$. Hence, for every $\alpha \in \mathbb{R}$, we have $e^{\alpha p} = \sum_{n=0}^{\infty} \frac{(\alpha p)^n}{n!} = \sum_{n=0}^{\infty} \frac{(\alpha p)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(\alpha p)^{(2n+1)}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n}}{(2n)!} + p \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{2n+1}}{(2n+1)!} = \cos\alpha + p \sin\alpha$. If $w \in \mathcal{H}$, then $e^w = e^{a_0 + q_t}$ where a_0 is the real part of w and q_t is a pure quaternion. If $0 \neq x \in \mathcal{H}$, then $\frac{x}{\|x\|}$ is a unit quaternion so that if $q_t \neq 0$, then $e^w = e^{a_0} e^{q_t} = e^{a_0} e^{\|q_t\| \frac{q_t}{\|q_t\|}} = e^{a_0}(\cos\|q_t\| + \frac{q_t}{\|q_t\|} \sin\|q_t\|)$. Observe that $e^{\boldsymbol{i}} = \cos 1 + \boldsymbol{i} \sin 1$, $e^{\boldsymbol{j}} = \cos 1 + \boldsymbol{j} \sin 1$, $e^{\boldsymbol{i}+\boldsymbol{j}} = \cos\sqrt{2} + \frac{\boldsymbol{i}+\boldsymbol{j}}{\sqrt{2}} \sin\sqrt{2}$ and $e^{\boldsymbol{i}} e^{\boldsymbol{j}} = (\cos 1 + \boldsymbol{i} \sin 1)(\cos 1 + \boldsymbol{j} \sin 1) = \cos^2 1 + \boldsymbol{j}(\cos 1 \sin 1) + \boldsymbol{i}(\cos 1 \sin 1) + \boldsymbol{k} \sin^2 1$. Hence, $e^{\boldsymbol{i}+\boldsymbol{j}} \neq e^{\boldsymbol{i}} e^{\boldsymbol{j}}$. Moreover, one checks that $e^{\boldsymbol{i}} e^{\boldsymbol{j}} \neq e^{\boldsymbol{j}} e^{\boldsymbol{i}}$.

Let S^3 be the set of all unit quaternions. Let $B_{a,b} : \mathcal{H} \to \mathcal{H}$ be given by $B_{a,b}(x) = axb$ where $a, b \in S^3$. Let G be the group of real linear transformation $L : \mathcal{H} \to \mathcal{H}$ satisfying

$$\langle L(e), L(d) \rangle_1 = \langle e, d \rangle_1 = Re(e\overline{d})$$

where e and $d \in \mathcal{H}$. Notice that for $x, y \in \mathcal{H}$, we have $Re(xy) = Re(yx)$. Since $\langle B_{a,b}(e), B_{a,b}(d) \rangle_1 = \langle aeb, adb \rangle_1 = Re((aeb)\overline{(adb)}) = Re(aeb\overline{b}\ \overline{d}\overline{a}) = Re(ae\overline{d}\overline{a}) = Re(e\overline{d}) = \langle e, d \rangle_1$, we have that $B_{a,b}$ is a member of $O(4)$. Looking at $e = e_1 + e_2 \boldsymbol{i} + e_3 \boldsymbol{j} + e_4 \boldsymbol{k} \in \mathcal{H}$ as a vector $\widehat{e} = [e_1\ e_2\ e_3\ e_4]^T \in \mathbb{R}^4$, $Re(e\overline{d}) = \left\langle \widehat{e}, \widehat{d} \right\rangle_2$ (the usual inner product), also we have $L(e) = A\widehat{e}$, for some $A \in M_4(\mathbb{R})$. One checks that $\langle L(e), d \rangle_1 = \left\langle A\widehat{e}, \widehat{d} \right\rangle_2 = \left\langle \widehat{e}, A^T \widehat{d} \right\rangle_2$. Hence, $L \in G$ if and only if $\left\langle A\widehat{e}, A\widehat{d} \right\rangle_2 = \left\langle \widehat{e}, A^T A\widehat{d} \right\rangle_2 = \left\langle \widehat{e}, \widehat{d} \right\rangle_2$ for every $\widehat{d}, \widehat{e} \in \mathbb{R}^4$, that is, if and only if A is a 4-by-4 real orthogonal matrix. Let $O(4)$ be the group of all 4-by-4 real orthogonal matrices. Then G as a group is isomorphic to $O(4)$. Let $SO(4)$ be the subgroup of $O(4)$ whose determinant is 1.

Let $Sp(2)$ be the group of 2-by-2 matrices $A \in M_2(\mathcal{H})$ such that $AA^* = I$. Then $Sp(2)$ is a compact symplectic group, the quaternionic analogue of the complex unitary group.

Let $y \in S^3$ be given. A *reflection* in S^3 about a hyperplane in $\mathcal{H}$ perpendicular to y is given by the linear mapping $f_y(x) = -y\overline{x}y = x - 2\,\mathrm{Re}(x\overline{y})y$ [1], and is represented by the Householder matrix $A = I - 2\widehat{y}\widehat{y}^T$. Let l, m, n and $v \in \mathbb{R}$ be given. To $y = l + \boldsymbol{i}m + \boldsymbol{j}n + \boldsymbol{k}v$, we associate a quaternionic matrix

$$Y = \begin{pmatrix} \boldsymbol{i}l + \boldsymbol{j}m + \boldsymbol{k}n & v \\ -v & -(\boldsymbol{i}l + \boldsymbol{j}m + \boldsymbol{k}n) \end{pmatrix}. \tag{1}$$

One checks that Y is unitary ($YY^* = Y^*Y = I$), that Y is skew-Hermitian ($Y^* = -Y$), and that $Y^2 = -I$. Let $PSp(2) = Sp(2)/(\pm I)$ be a projective symplectic group. Every element of $O(4)$ is a product of Householder matrices [3, Theorem 1]. Hence, we say that $O(4)$ is generated by the set of reflections f_y, so that the correspondence $y \mapsto Y$ defined by equation (1) may be extended to an injective group homomorphism $\xi : O(4) \to PSp(2)$ such that $\xi(f_y) = [Y]$, the equivalence class of Y in $PSp(2)$ [2]. The mapping ξ may be shown to be continuous, and hence, differentiable. We evaluate the differential of ξ.

2 Quaternionic Matrices

Notice that $B_{a,b}$ is an orthogonal transformation. If $Re(a) = Re(b)$, then there exists $p \in S^3$ such that $ap = pb$ [1]. Moreover, because $||b|| = 1$, we also have $b^{-1} = \bar{b}$, so that $Re(b^{-1}) = Re(b)$ as well. Hence, there exists $q \in S^3$, such that $aq = qb^{-1}$. Let $z = ap = pb$, then $B_{a,b}(x) = f_{ap} \circ f_p(x) = f_z \circ f_p(x)$ [1]. If $Re(a) \neq Re(b)$, then we express $B_{a,b}(x)$ as a product of two rotations given by $B_{a,b}(x) = B_{a,1} \circ B_{1,b}(x)$. We express a as a power of a pure quaternion s, say $a = s^t$ where $t \in \mathbb{R}$. Then $B_{s^t,1}(x) = B_{s^{\frac{t}{2}},s^{\frac{t}{2}}} \circ B_{s^{\frac{t}{2}},s^{\frac{-t}{2}}}(x)$. Similarly, we follow the same method to compute $B_{1,b}(x)$. Notice that each $B_{a,b}(x)$ is a product of two (or an even number of) reflections. Hence, it is a member of $SO(4)$ because the determinant of reflection is -1.

We compute $B_{\boldsymbol{i}^t,1}(x)$, with $t \in \mathbb{R}$. Since $Re(\boldsymbol{i}^t) \neq Re(1)$, we write $B_{\boldsymbol{i}^t,1}(x)$ as a product of two rotations given by $B_{\boldsymbol{i}^t,1}(x) = B_{\boldsymbol{i}^{\frac{t}{2}},\boldsymbol{i}^{\frac{t}{2}}} \circ B_{\boldsymbol{i}^{\frac{t}{2}},\boldsymbol{i}^{\frac{-t}{2}}}(x)$. Now we write each of $B_{\boldsymbol{i}^{\frac{t}{2}},\boldsymbol{i}^{\frac{t}{2}}}$ and $B_{\boldsymbol{i}^{\frac{t}{2}},\boldsymbol{i}^{\frac{-t}{2}}}$ as a product of two reflections. The real part of $\boldsymbol{i}^{\frac{t}{2}}$ and $\boldsymbol{i}^{\frac{-t}{2}}$ are same. Observe that $B_{\boldsymbol{i}^{\frac{t}{2}},\boldsymbol{i}^{\frac{t}{2}}} = f_{\boldsymbol{i}^{\frac{t}{2}}} \circ f_1$ and $B_{\boldsymbol{i}^{\frac{t}{2}},\boldsymbol{i}^{\frac{-t}{2}}} = f_{\boldsymbol{i}^{\frac{t}{2}}\boldsymbol{j}} \circ f_{\boldsymbol{j}}$. Hence $B_{\boldsymbol{i}^t,1}(x) = f_{\boldsymbol{i}^{\frac{t}{2}}} \circ f_1 \circ f_{\boldsymbol{i}^{\frac{t}{2}}\boldsymbol{j}} \circ f_{\boldsymbol{j}}$.

If $p \in S^3$ is pure, then $p = e^{p\frac{\pi}{2}}$ and $p^t = e^{p\frac{\pi}{2}t} = \cos\frac{\pi t}{2} + p \sin\frac{\pi t}{2}$. Hence, $\boldsymbol{i}^{\frac{t}{2}} = \cos\frac{\pi}{4}t + \boldsymbol{i}\sin\frac{\pi}{4}t$. Let $y_1 = \boldsymbol{i}^{\frac{t}{2}}$. Then f_{y_1} (refer to equation (1)) is associated with

$$Y_1(t) = \begin{pmatrix} \boldsymbol{i}\cos\frac{\pi}{4}t + \boldsymbol{j}\sin\frac{\pi}{4}t & 0 \\ 0 & -(\boldsymbol{i}\cos\frac{\pi}{4}t + \boldsymbol{j}\sin\frac{\pi}{4}t) \end{pmatrix}.$$

Note that

$$Y_1'(t) = \frac{\pi}{4}\begin{pmatrix} -\boldsymbol{i}\sin\frac{\pi}{4}t + \boldsymbol{j}\cos\frac{\pi}{4}t & 0 \\ 0 & \boldsymbol{i}\sin\frac{\pi}{4}t - \boldsymbol{j}\cos\frac{\pi}{4}t \end{pmatrix}.$$

Let $y_2 = 1$. Then f_{y_2} is associated with

$$Y_2(t) = \begin{pmatrix} \boldsymbol{i} & 0 \\ 0 & -\boldsymbol{i} \end{pmatrix}.$$

and ,

$$Y_2'(t) = 0.$$

Let $y_3 = \boldsymbol{i}^{\frac{t}{2}}\boldsymbol{j}$. Then f_{y_3} is associated with

$$Y_3(t) = \begin{pmatrix} \boldsymbol{k}\cos\frac{\pi}{4}t & \sin\frac{\pi}{4}t \\ -\sin\frac{\pi}{4}t & -\boldsymbol{k}\cos\frac{\pi}{4}t \end{pmatrix}.$$

Thus

$$Y_3'(t) = \frac{\pi}{4}\begin{pmatrix} -\boldsymbol{k}\sin\frac{\pi}{4}t & \cos\frac{\pi}{4}t \\ -\cos\frac{\pi}{4}t & \boldsymbol{k}\sin\frac{\pi}{4}t \end{pmatrix}.$$

Let $y_4 = \boldsymbol{j}$. Then f_{y_4} is associated with

$$Y_4(t) = \begin{pmatrix} \boldsymbol{k} & 0 \\ 0 & -\boldsymbol{k} \end{pmatrix},$$

and

$$Y_4'(t) = 0.$$

We now compute the differential dξ using the product rule without change in the order of the factors.

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \xi(B_{\boldsymbol{i}^t,1}) &= \left.\frac{d}{dt}\right|_{t=0} \xi(f_{\boldsymbol{i}^{\frac{t}{2}}} \circ f_1 \circ f_{\boldsymbol{i}^{\frac{t}{2}}\boldsymbol{j}} \circ f_{\boldsymbol{j}}) \\ &= \left[\left.\frac{d}{dt}\right|_{t=0} \xi(f_{\boldsymbol{i}^{\frac{t}{2}}})\right] \xi(f_1 \circ f_{\boldsymbol{i}^{\frac{t}{2}}\boldsymbol{j}} \circ f_{\boldsymbol{j}}) \\ &\quad + \xi\left(f_{\boldsymbol{i}^{\frac{t}{2}}}\right) \left[\left.\frac{d}{dt}\right|_{t=0} \xi(f_1)\right] \xi\left(f_{\boldsymbol{i}^{\frac{t}{2}}\boldsymbol{j}} \circ f_{\boldsymbol{j}}\right) \\ &\quad + \xi\left(f_{\boldsymbol{i}^{\frac{t}{2}}} \circ f_1\right) \left[\left.\frac{d}{dt}\right|_{t=0} \xi(f_{\boldsymbol{i}^{\frac{t}{2}}\boldsymbol{j}})\right] \xi\left(f_{\boldsymbol{j}}\right) \end{aligned}$$

$$+\xi\left(f_{\boldsymbol{i}^{\frac{t}{2}}}\circ f_1\circ f_{\boldsymbol{i}^{\frac{t}{2}}\boldsymbol{j}}\right)\left[\frac{d}{dt}\bigg|_{t=0}\xi\left(f_{\boldsymbol{j}}\right)\right]$$
$$= \frac{\pi}{4}\boldsymbol{k}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$$

Similarly, we have

$$\frac{d}{dt}\bigg|_{t=0}\xi(B_{\boldsymbol{j}^t,1}) = -\frac{\pi}{4}\boldsymbol{j}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix},$$

and

$$\frac{d}{dt}|_{t=0}\xi(B_{\boldsymbol{k}^t,1}) = \frac{\pi}{4}\boldsymbol{i}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

We now calculate $\frac{d}{dt}|_{t=0}\xi(B_{1,\boldsymbol{i}^t})$. We write $B_{1,\boldsymbol{i}^{\frac{t}{2}}}(x)$ as a product of reflections. $B_{1,\boldsymbol{i}^{\frac{t}{2}}}(x) = B_{\boldsymbol{i}^{-\frac{t}{2}}\boldsymbol{i}^{\frac{t}{2}}}\circ B_{\boldsymbol{i}^{\frac{t}{2}}\boldsymbol{i}^{\frac{t}{2}}}(x) = f_{\boldsymbol{i}^{-\frac{t}{2}}\boldsymbol{j}}\circ f_{\boldsymbol{j}}\circ f_{\boldsymbol{i}^{\frac{t}{2}}}\circ f_1$. Then we associate each f_y with respective Y and Y'. Now, we have

$$\frac{d}{dt}|_{t=0}\xi(B_{1,\boldsymbol{i}^t}) = -\frac{\pi}{4}\boldsymbol{k}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.$$

Similarly,

$$\frac{d}{dt}|_{t=0}\xi(B_{1,\boldsymbol{j}^t}) = -\frac{\pi}{4}\boldsymbol{j}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix},$$

and

$$\frac{d}{dt}|_{t=0}\xi(B_{1,\boldsymbol{k}^t}) = -\frac{\pi}{4}\boldsymbol{i}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}.$$

References

[1] Coxeter, H.S.M., Quaternions and reflections, *Amer. Math. Monthly*, **53** (1946), 136–146.

[2] Canlubo, C. and Reyes, E., Reflections on S^3 and Quaternionic Möbius Transformations, *J. Lie Theory*, **22** (2012), 839-844.

[3] F. Uhlig, Constructive ways for generating (generalized) real orthogonal matrices as products of (generalized) symmetries, *Linear Algebra Appl.*, **332–334** (2001) 459–467.