7.4 – The Other Trig Functions

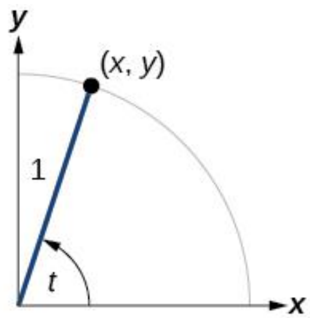
Learning Objectives

In this section you will:

- Find exact values of the trigonometric functions secant, cosecant, tangent, and cotangent of $\frac{\pi}{3}, \frac{\pi}{4}$, and $\frac{\pi}{6}$.
- Use reference angles to evaluate the trigonometric functions secant, tangent, and cotangent.
- Use properties of even and odd trigonometric functions.
- Recognize and use fundamental identities.
- Evaluate trigonometric functions with a calculator.

Finding Exact Values

As with the sine and cosine, we can use the(x,y)coordinates to find the other functions.



T

ANGENT, SECANT, COSECANT, AND COTANGENT FUNCTIONS

If t is a real number and(x , y) is a point where the terminal side of an angle of t radians intercepts the unit circle, then

tan t = $\frac{y}{x}, x \neq 0$ sec t = $\frac{1}{x}, x \neq 0$

csc t = $\frac{1}{y}, y \neq 0$ cot t = $\frac{x}{y}, y \neq 0$

Example **Finding Trigonometric Functions from a Point on the Unit Circle**

The point on the unit circle, as shown in **Figure 2**. Find sin *t*, cos *t*, tan *t*, sec *t*, csc *t*, and cot *t*.

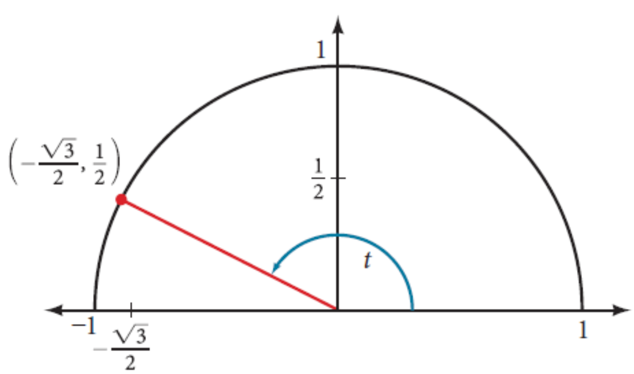
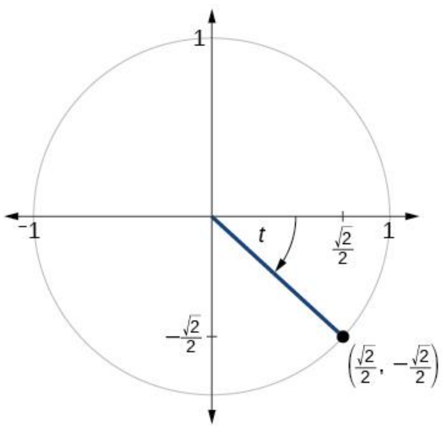


Figure 2

Answer: sin t = $\frac{1}{2}$, Cos t = $-\frac{\sqrt{3}}{2}$, tan t = $\frac{Sin\ t}{Cos\ t} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$.

Csc t = 2 (flip sin t), Sec t = $-\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$ (flip cos t), Cot t = - $\sqrt{3}$ (flip tan t)

Try It: The point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ is on the unit circle, as shown in Figure. Find $sin\ t$, $cos\ t$, $tan\ t$, $sec\ t$, $csc\ t$, and $cot\ t$.

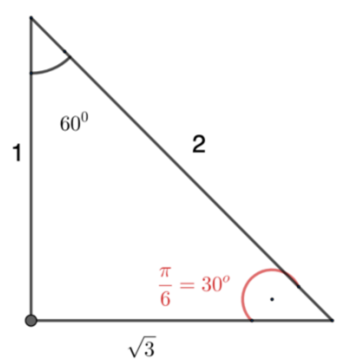


Answer: sin t = $-\frac{\sqrt{2}}{2}$, Cos t = $\frac{\sqrt{2}}{2}$, tan t = $\frac{Sin\ t}{Cos\ t} = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$.

Csc t = - $\frac{1}{\sqrt{2}}$ (flip sin t), Sec t = $\frac{1}{\sqrt{2}}$ (flip sec t), Cot t = - 1 (flip tan t).

Example **Finding the Trigonometric Functions of an Angle**

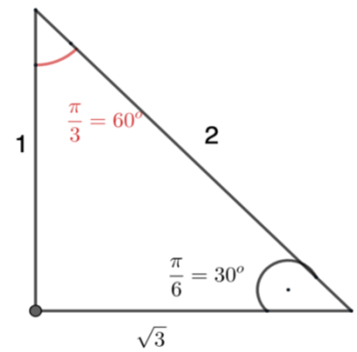
Find sin *t*, cos *t*, tan *t*, sec *t*, csc *t*, and cot *t* when *t* = *π*/6 .



Sint ($\pi/6$) = $\frac{1}{2}$, Cos ($\pi/6$)= $\frac{\sqrt{3}}{2}$, Tan ($\pi/6$)= $\frac{1}{\sqrt{3}}$ = $\frac{\sqrt{3}}{3}$

Csc ($\pi/6$) =2, Sec ($\pi/6$) = $\frac{2}{\sqrt{3}}$ = $\frac{2\sqrt{3}}{3}$, Cot ($\pi/6$) = $\sqrt{3}$.

Try It: Find sin t, cos t, tan t, sec t, csc t, and cot t when t = $\frac{\pi}{3}$



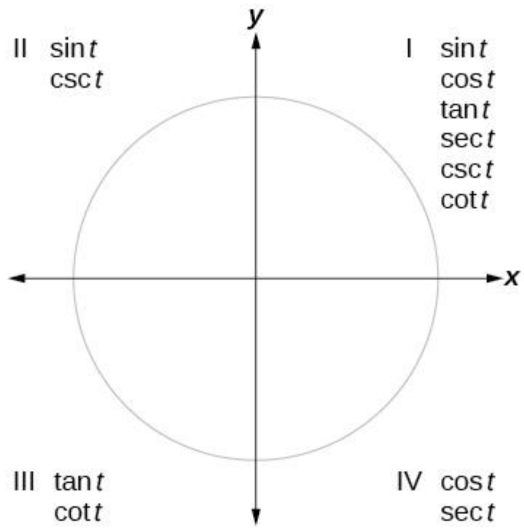
Sint ($\pi/3$) =$\frac{\sqrt{3}}{2}$, Cos ($\pi/3$)= $\frac{1}{2}$, Tan ($\pi/3$)= $\sqrt{3}$.
Csc ($\pi/3$) =$\frac{2}{\sqrt{3}}$=$\frac{2\sqrt{3}}{3}$, Sec ($\pi/3$) =2, Cot ($\pi/3$) =$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$.

Because we know the sine and cosine values for the common first-quadrant angles, we can find the other function values for those angles as well.

Angle	0	$\frac{\pi}{6}$, or $30°$	$\frac{\pi}{4}$, or $45°$	$\frac{\pi}{3}$, or $60°$	$\frac{\pi}{2}$, or $90°$
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
Tangent	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined
Secant	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	Undefined
Cosecant	Undefined	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1
Cotangent	Undefined	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Using Reference Angles to Evaluate Secant, Cosecant, and Cotangent

We can evaluate trigonometric functions of angles outside the first quadrant using reference angles as we have already done with the sine and cosine functions.



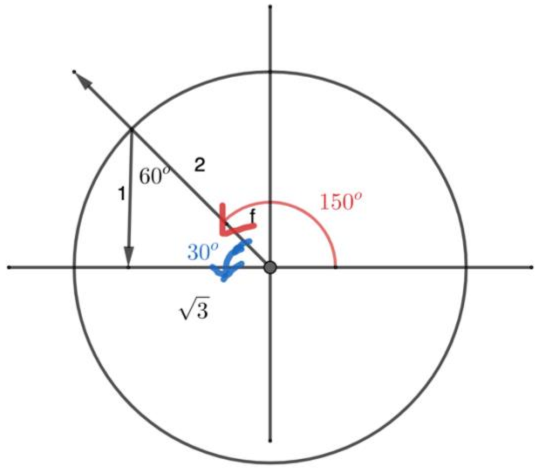
To help us remember which of the six trigonometric functions are positive in each quadrant, we can use the mnemonic phrase "**A Smart Trig Class**." Each of the four words in the phrase corresponds to one of the four quadrants, starting with quadrant I and rotating counterclockwise.
In quadrant I, which is "**A**," **a**ll of the six trigonometric functions are positive.
In quadrant II, "**S**mart," only **s**ine and its reciprocal function, cosecant, are positive.
In quadrant III, "**T**rig," only **t**angent and its reciprocal function, cotangent, are positive.
Finally, in quadrant IV, "**C**lass," only **c**osine and its reciprocal function, secant, are positive.

How To:
Given an angle not in the first quadrant, use reference angles to find all six trigonometric functions.

1. Measure the angle formed by the terminal side of the given angle and the horizontal axis. This is the reference angle.
2. Evaluate the function at the reference angle.
3. Observe the quadrant where the terminal side of the original angle is located. Based on the quadrant, determine whether the output is positive or negative.

Example **Using Reference Angles to Find Trigonometric Functions**
Use reference angles to find all six trigonometric functions of $5\pi/6$.

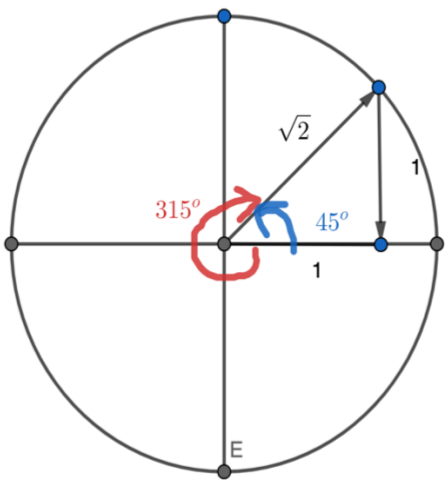
$$You\ can\ first\ convert\ \frac{5\pi}{6}\ into\ degrees\ first\ if\ your\ prefer.(\frac{5\pi}{6}$$
$$= 150°).Clearly, this\ lies\ in\ 2nd\ quadrant. So\ the\ reference\ angle\ is\ 180 - 150$$
$$= 30°\ (\pi - \frac{5\pi}{6} = \frac{\pi}{6})$$



$$Sin\,(\frac{5\pi}{6}) = Sin\,(\frac{\pi}{6}) = \frac{1}{2}, Cos\,(\frac{5\pi}{6}) = -Cos\,(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}, Tan\,(\frac{5\pi}{6}) = -\frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

$$Csc\,(\frac{5\pi}{6}) = 2, Sec\,(\frac{5\pi}{6}) = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}, Cot\,(\frac{5\pi}{6}) = -\sqrt{3}.$$

Try It: Use reference angles to find all six trigonometric functions of $-\frac{7\pi}{4}$.

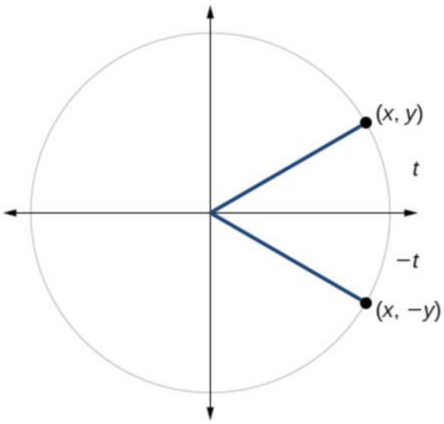


$\frac{-7\pi}{4}$ $(315\ degree)$ is a negative angle so we move clockwise direction. It lies in first quadrant (so all values will be positive), so the reference angle is 360-315= 45°.

$$Sin\left(\frac{-7\pi}{4}\right) = Sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, Cos\left(\frac{-7\pi}{4}\right) = Cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, Tan\left(\frac{-7\pi}{4}\right) = 1.$$
$$Csc\left(\frac{-7\pi}{4}\right) = \sqrt{2}, \quad Sec\left(\frac{-7\pi}{4}\right) = \sqrt{2}, \quad Cot\left(\frac{-7\pi}{4}\right) = 1.$$

Using Even and Odd Trig Functions

We can test whether a trigonometric function is even or odd by drawing a unit circle with a positive and a negative angle, as in the figure below. We can test each of the six trigonometric functions in this fashion. The results are shown in Table 2.



$\sin t = y$ $\sin(-t) = -y$ $\sin t \neq \sin(-t)$	$\cos t = x$ $\cos(-t) = x$ $\cos t = \cos(-t)$	$\tan(t) = \frac{y}{x}$ $\tan(-t) = -\frac{y}{x}$ $\tan t \neq \tan(-t)$
$\sec t = \frac{1}{x}$ $\sec(-t) = \frac{1}{x}$ $\sec t = \sec(-t)$	$\csc t = \frac{1}{y}$ $\csc(-t) = \frac{1}{-y}$ $\csc t \neq \csc(-t)$	$\cot t = \frac{x}{y}$ $\cot(-t) = \frac{x}{-y}$ $\cot t \neq \cot(-t)$

Table 2

A GENERAL NOTE: EVEN AND ODD TRIGONOMETRIC FUNCTIONS
An even function is one in which f(−x) = f(x).
An odd function is one in which f(−x) = −f(x).

Cosine and secant are even:
cos(−t) = cos t
sec(−t) = sec t

Sine, tangent, cosecant, and cotangent are odd:
sin(−t) = −sin t

tan(−t) = −tan t
csc(−t) = −csc t
cot(−t) = −cot t

Example **Using Even and Odd Properties of Trigonometric Functions**
If the secant of angle *t* is 2, what is the secant of −*t* ?
Sect (t)=2 implies Sec (-t) =2.

Try It: If the cotangent of angle t is √3, what is the cotangent of−t?
Cot (t) = $\sqrt{3}$ implies Cot (-t) = $-\sqrt{3}$

Recognizing and Using Fundamental Identities
Identities are statements that are true for all values of the input on which they are defined.

A GENERAL NOTE: FUNDAMENTAL IDENTITIES
We can derive some useful **identities** from the six trigonometric functions. The other four trigonometric functions can be related back to the sine and cosine functions using these basic relationships:

$$\tan t = \frac{\sin t}{\cos t}$$
$$\sec t = \frac{1}{\cos t}$$
$$\csc t = \frac{1}{\sin t}$$
$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}$$

Example **Using Identities to Simplify Trigonometric Expressions**
Simplify $\frac{sect.}{\tan t}$
$$\frac{Sec\ t}{\tan t} = Sec\ t * \frac{1}{\tan t} = \frac{1}{Cos\ t} * \frac{Cos\ t}{Sin\ t} = \frac{1}{Sin\ t} = Csc\ t.$$

Try It: Simplify $(tan\ t)(cos\ t)$.
$$\tan t\ Cos\ t = \frac{\text{Sin}\ t}{Cos\ t} * Cos\ t = Sin\ t$$

Example **Using Identities to Evaluate Trigonometric Functions**
Given cos $(5\pi/6) = -\sqrt{3}/2$, evaluate sec$(5\pi/6)$.

Cos and Sec are reciprocal of each other, so $Sec\left(\frac{5\pi}{6}\right) = -\frac{2}{\sqrt{3}} = -\frac{2}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$.

Example **Using Identities to Evaluate Trigonometric Functions**
Given sin (45°) = $\sqrt{2}/2$, cos (45°) = $\sqrt{2}/2$, evaluate tan(45°).

Tan(45°)=$\frac{Sin45°}{Cos\ 45°} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1.$

Alternate forms of the Pythagorean Theorem

We can use these fundamental identities to derive alternative forms of the Pythagorean Identity.

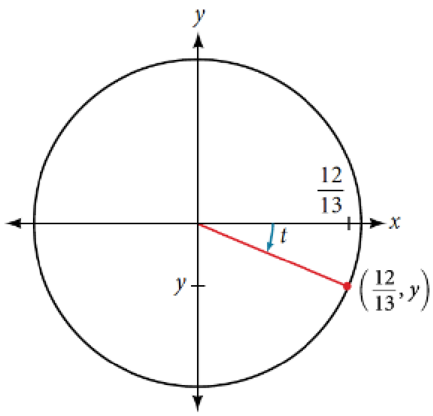
PYTHAGOREAN IDENTITY
The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1$$

ALTERNATE FORMS OF THE PYTHAGOREAN IDENTITY

$$1 + \tan^2 t = \sec^2 t$$
$$\cot^2 t + 1 = \csc^2 t$$

Example **Using Identities to Relate Trigonometric Functions**
If cos(t) = 12/13 and t is in quadrant IV, as shown in Figure 8, find the values of the other five trigonometric functions.



We use $\cos^2 t + \sin^2 t = 1$.

$$\left(\frac{12}{13}\right)^2 + Sin^2(t) = 1$$

$$\frac{144}{169} + Sin^2(t) = 1$$

$Sin^2(t) = 1 - \frac{144}{169}$

$Sin^2(t) = \frac{169}{169} - \frac{144}{169}$

$Sin^2(t) = \frac{169-144}{169}.$

$Sin^2(t) = \frac{25}{144}$

$Sin^2(t) = \pm\sqrt{\frac{25}{144}} = \pm\frac{5}{12}$

Since this is 4th quadrant, y=$-\frac{5}{12} = Sin\ (t)$.

$$Sin\ (t) = -\frac{5}{12}, \quad Cos(t) = \frac{12}{13}, Tan(t) = \frac{-5/_{12}}{12/_{13}} = \frac{-5}{12} * \frac{12}{13} = \frac{-5}{12}$$

$$Csc(t) = -\frac{12}{5}, \quad Sec\ (t) = \frac{13}{12}, \quad Cot(t) = -\frac{12}{5}.$$

Try It: If sec(t) = $-\frac{17}{8}$ and 0 < t < π, find the values of the other five functions.

Sec(t) and Cos(t) are reciprocal. $Cos(t) = -\frac{8}{17}$. So the angle must lie in 2nd quadrant because x is negative and 0 < t < π

We use $\cos^2 t + \sin^2 t = 1$.

$\left(\frac{-8}{17}\right)^2 + Sin^2(t) = 1$

$\frac{64}{289} + Sin^2(t) = 1$

$Sin^2(t) = 1 - \frac{64}{289}$

$Sin^2(t) = \frac{289}{289} - \frac{64}{289}$

$Sin^2(t) = \frac{289-64}{289}.$

$Sin^2(t) = \frac{264}{289}$

$Sin(t) = \pm\sqrt{\frac{264}{289}} = \pm\frac{\sqrt{264}}{17} = \pm\frac{2\sqrt{66}}{17}$

Since this is in 2nd quadrant, we have $Sin\ (t) = \frac{2\sqrt{66}}{17}$

$$Sin\ (t) = \frac{2\sqrt{66}}{17}, \quad Cos(t) = \frac{-8}{17}, \quad Tan(t) = \frac{2\sqrt{66}/_{17}}{-8/_{17}} = -\frac{2\sqrt{66}}{17} * \frac{17}{8} = \frac{-\sqrt{66}}{4}$$

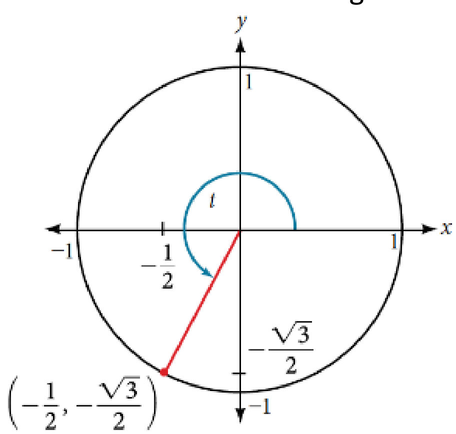
$$Csc(t) = \frac{17}{2\sqrt{66}} * \frac{\sqrt{66}}{\sqrt{66}} = \frac{17\sqrt{66}}{132}, \quad Sec\ (t) = -\frac{17}{8}, \quad Cot(t) = -\frac{4}{\sqrt{66}} * \frac{\sqrt{66}}{\sqrt{66}} = \frac{2\sqrt{66}}{33}.$$

A function that repeats its values in regular intervals is known as a **periodic** function. The trigonometric functions are periodic.

A period is the shortest interval over which a function completes one full cycle.

A GENERAL NOTE: PERIOD OF A FUNCTION
The **period** P of a repeating function f is the number representing the interval such that f(x + P) = f(x) for any value of x.
The period of the cosine, sine, secant, and cosecant functions is 2π.
The period of the tangent and cotangent functions is π.

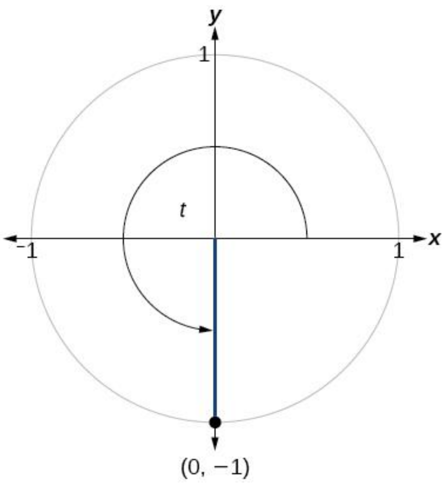
Example **Finding the Values of Trigonometric Functions**
Find the values of the six trigonometric functions of angle *t* based on **Figure 9**.



$$Sin\ (t) = -\frac{\sqrt{3}}{2}, \qquad Cos(t) = -\frac{1}{2}, Tan(t) = \frac{-\sqrt{3}/2}{-1/2} = -\frac{\sqrt{3}}{2} * -\frac{2}{1} = \sqrt{3}$$

$$Csc(t) = \frac{-2}{\sqrt{3}} * \frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}, \qquad Sec\ (t) = -2, \qquad Cot(t) = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Try It:
Find the values of the six trigonometric functions of angle t based on Figure.



$$Sin\ (t) = -1, \quad Cos(t) = 0\ , Tan(t) = \frac{-1}{0} = \text{undefined}.$$
$$Csc(t) = -1, \quad Sec\ (t) = \frac{1}{0} = undefined, \quad Cot(t) = \frac{0}{-1} = 0.$$

Example **Finding the Value of Trigonometric Functions**
If sin(*t*) = − √3/2 and cos(*t*) = 1/2 , find sec(*t*), csc(*t*), tan(*t*), cot(*t*).

$$Sec(t) = \frac{1}{Cos(t)} = 2\ , Csc(t) = \frac{1}{\text{Sin}(t)} = \frac{2}{-\sqrt{3}}$$
$$Tan(t) = \frac{Sin(t)}{Cos(t)} = \frac{-\sqrt{3}/2}{1/2} = \frac{-\sqrt{3}}{2} * \frac{2}{1} = -\sqrt{3}, Cot\ (t) = \frac{1}{Tan(t)} = \frac{1}{-\sqrt{3}} * \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

Evaluating Trig Functions with a Calculator

To evaluate trigonometric functions of other angles, we use a scientific or graphing calculator or computer software. If the calculator has a degree mode and a radian mode, confirm the correct mode is chosen before making a calculation.

For the reciprocal functions, there may not be any dedicated keys that say CSC, SEC, or COT. In that case, the function must be evaluated as the reciprocal of a sine, cosine, or tangent.

How To:

Given an angle measure in radians, use a calculator to find the cosecant.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Enter: 1/
3. Press the SIN key.
4. Enter the value of the angle inside parentheses.

5. Press the = [ENTER] key.

Example **Evaluating Using Technology**

Evaluate the cosecant of $\frac{5\pi}{7}$

Evaluate sec(125°)

Evaluate the tan(312°)

Try It: Evauate the cotangent of $-\frac{\pi}{8}$

Use the table to find the EXACT value.			
1. $\csc 135°$	2. $\sin \pi$	3. $\cot\left(-\frac{11\pi}{6}\right)$	4. $\sec(-90°)$
5. $\sin 315°$	6. $\sin \frac{7\pi}{6}$	7. $\tan 765°$	8. $\cot -\frac{19\pi}{6}$
9. $\csc -135°$	10. $\cos -900°$	11. $\sec -690°$	12. $\tan \frac{11\pi}{6}$
Use the calculator to find the APPROXIMATE value of each.			
13. $\csc 80°$	14. $\cot 15°$	15. $\sec 40°$	16. $\sin 51°$
17. $\sin \frac{\pi}{18}$	18. $\tan \frac{7\pi}{18}$	19. $\cot \frac{23\pi}{90}$	20. $\cot \frac{\pi}{5}$
21. $\csc \frac{\pi}{18}$	22. $\sec 115°$	23. $\csc \frac{5\pi}{18}$	24. $\sin 1.2$
Use the table to find each angle where $0° \leq \theta \leq 360°$.			
25. $\cos\theta = -\frac{1}{2}$	26. $\csc\theta = \frac{2\sqrt{3}}{3}$	27. $\tan\theta =$ undefined	28. $\sin\theta = -\frac{\sqrt{2}}{2}$
29. $\csc\theta = 1$	30. $\sec\theta = \sqrt{2}$	31. $\tan\theta = 0$	32. $\sin\theta = 0$