7.3 – The Unit Circle

Learning Objectives

In this section you will:

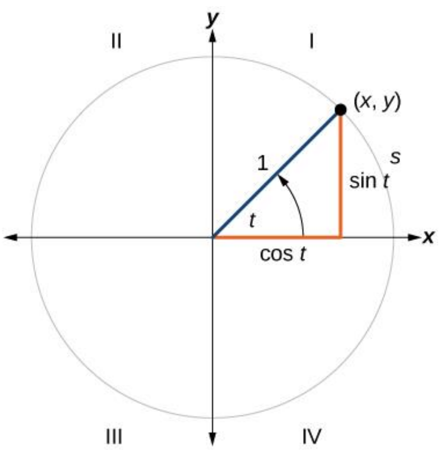
- Find function values for the sine and cosine of

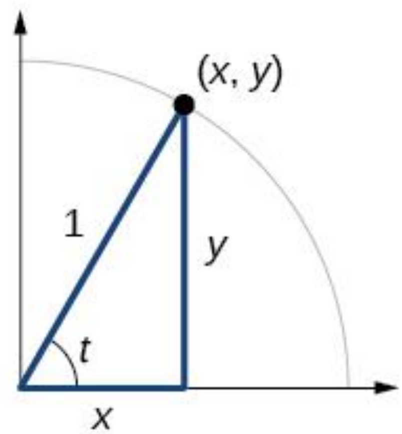
$$30° \text{ or } \left(\frac{\pi}{6}\right), 45° \text{ or } \left(\frac{\pi}{4}\right),\ and\ 60° \text{ or } \left(\frac{\pi}{3}\right).$$

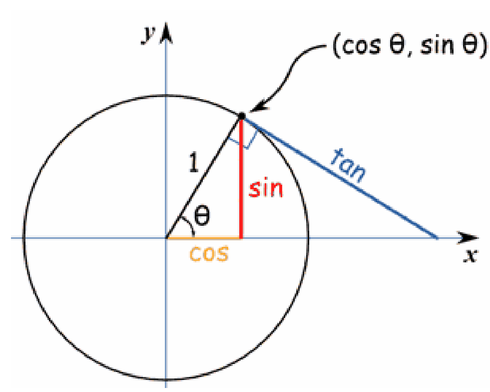
- Identify the domain and range of sine and cosine functions.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Finding Trig Functions Using The Unit Circle

For any angle *t*, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x, y).The coordinates *x* and *y* will be the outputs of the trigonometric functions *f*(*t*)=cos *t* and *f*(*t*)=sin *t*, respectively. This means *x*=cos *t* and *y*=sin *t*.







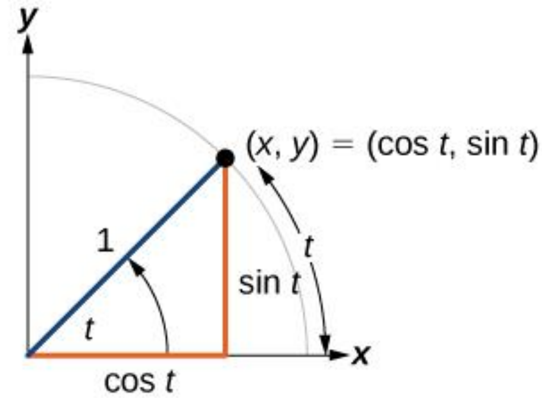
UNIT CIRCLE

A **unit circle** has a center at (0,0) and radius 1. In a unit circle, the length of the intercepted arc is equal to the radian measure of the central angle t.

Let (x , y) be the endpoint on the unit circle of an arc of **arc length** s. The (x , y) coordinates of this point can be described as functions of the angle.

Defining Sine and Cosine Functions from the Unit Circle

Like all functions, the sine function has an input and an output. Its input is the measure of the _______________, its output is the ___-coordinate of the corresponding point on the unit circle.
The cosine function of an angle *t* equals the *x*-value of the endpoint on the unit circle of an arc of ______________ *t*. In the figure, cosine is equal to _____.



Because it is understood that sine and cosine are functions, we do not always need to write them with parentheses: sin t is the same as sin(t) and cos t is the same as cos(t). Likewise, $\cos^2 t$ is a commonly used shorthand notation for$(\cos(t))^2$.Be aware that many calculators and computers do not recognize the shorthand notation. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

A GENERAL NOTE: SINE AND COSINE FUNCTIONS
If t is a real number and a point (x, y) on the unit circle corresponds to a central angle t, then

$$cos\ t = x$$
$$sin\ t = y$$

How To:
Given a point $P\ (x, y)$ on the unit circle corresponding to an angle of t, **find the sine and cosine.**

1. The sine of t is equal to the *y*-coordinate of point P: $sin\ t = y$.
2. The cosine of t is equal to the *x*-coordinate of point P: $cos\ t = x$.

Example **Finding Function Values for Sine and Cosine**
Point *P* is a point on the unit circle corresponding to an angle of t, as shown in **Figure 4.** Find $cos(t)$ and $sin(t)$.

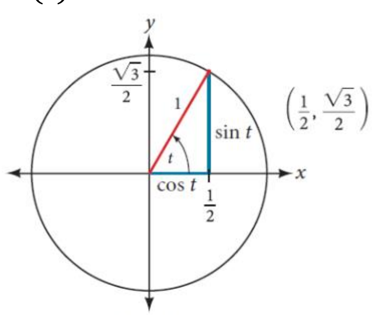
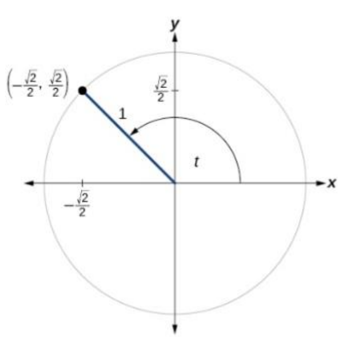
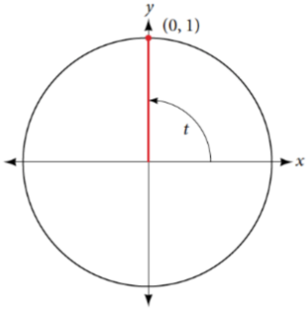


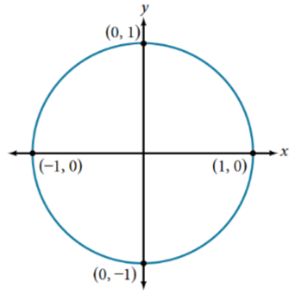
Figure 4

Try It: A certain angle t corresponds to a point on the unit circle at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ as shown in Figure. Find $cos\ t$ and $sin\ t$.



For quadrantral angles, the corresponding point on the unit circle falls on the *x*- or *y*-axis. In that case, we can easily calculate cosine and sine from the values of *x* and *y*.





Example Calculating Sines and Cosines along an Axis

Find cos(90°) and sin(90°).

Try It: Find cosine and sine of the angle π.

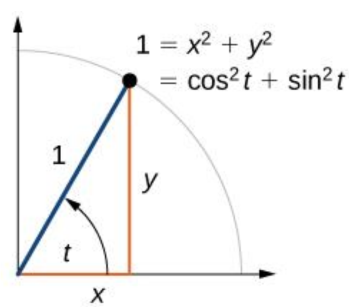
From the given information, find the quadrant in which the terminal point determined by t lies. Input I, II, III or IV.

a. Sin (t) < 0 and Cos (t) < 0, Quadrant______
b. Sin (t) > 0 and Cos (t) < 0, Quadrant______
c. Sin (t) > 0 and Cos (t) > 0, Quadrant______
d. Sin (t) < 0 and Cos (t) > 0, Quadrant________

PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any real number t,

$$cos^2\, t\; +\; sin^2\, t\; =\; 1$$



How To:

Given the sine of some angle t **and its quadrant location, find the cosine of** t.

1. Substitute the known value of sin t into the Pythagorean Identity.

2. Solve for cos t.
3. Choose the solution with the appropriate sign for the *x*-values in the quadrant where t is located

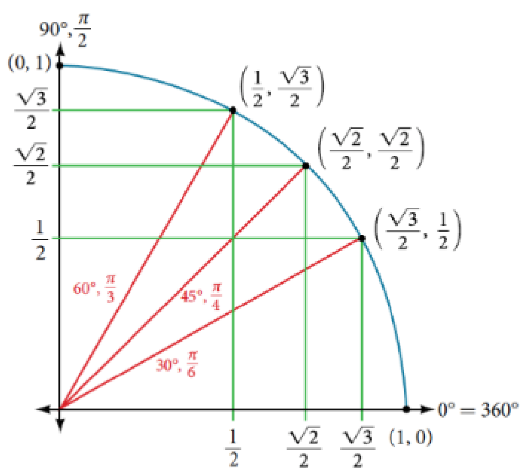
Example **Finding a Cosine from a Sine or a Sine from a Cosine**
If $sin(t) = 3/7$ and t is in the second quadrant, find $cos(t)$.
Note that

Try It: If cos(t) = $\frac{24}{25}$ and t is in the fourth quadrant, find sin(t).

Example: The point P is on the unit circle. If y-co-ordinate of P is -4/5 , and P is in quadrant III, then find x.

Finding Sine and Cosine of Special Angles

The image shows the ordered pairs of the cosine and sine values for all of the most commonly encountered angles in the first quadrant of the unit circle.
Table 1 summarizes these values.



Angle	0	$\frac{\pi}{6}$, or 30°	$\frac{\pi}{4}$, or 45°	$\frac{\pi}{3}$, or 60°	$\frac{\pi}{2}$, or 90°
Cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
Sine	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Note: Sine and cosine of the quadrantal and special angles will require an exact answer.
Using your calculator for these angles is not the proper approach.
Use the side length relationship with s = 1.

Example **Evaluate with Sine and Cosine of Special Angles**
Evaluate cos(45°).

Example **Evaluate with Sine and Cosine of Special Angles**
Evaluate sin($\pi/6$).

Example **Evaluate with Sine and Cosine of Special Angles**
Evaluate cos(60°).

Try It: Evaluate $\sin\left(\frac{\pi}{3}\right)$

Example **Evaluate with Sine and Cosine of Special Angles**
Evaluate $cos(\pi/4)sin(\pi/6)$.

Most calculators can be set into “degree” or “radian” mode, which tells the calculator the units for the input value.

How To:
Given an angle in radians, use a graphing calculator to find the cosine.

1. If the calculator has degree mode and radian mode, set it to radian mode.
2. Press the COS key.
3. Enter the radian value of the angle and press the close-parentheses key ")".
4. Press ENTER.

Be aware: We only use a calculator to find the cosine and sine of angles ***other than*** quadrantal and the special angles.

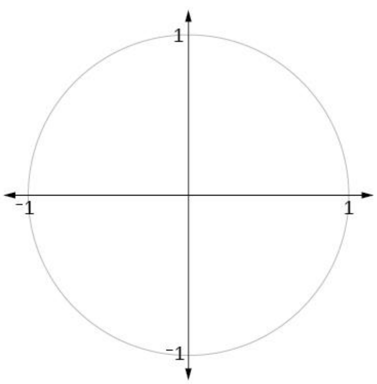
Example **Using a Graphing Calculator to Find Sine and Cosine**

Evaluate $\cos(7\pi/5)$.

Evaluate $\sin(114°)$

Evaluate $\sin(\pi/5)$.

Identifying Domain and Range of Sine and Cosine Functions



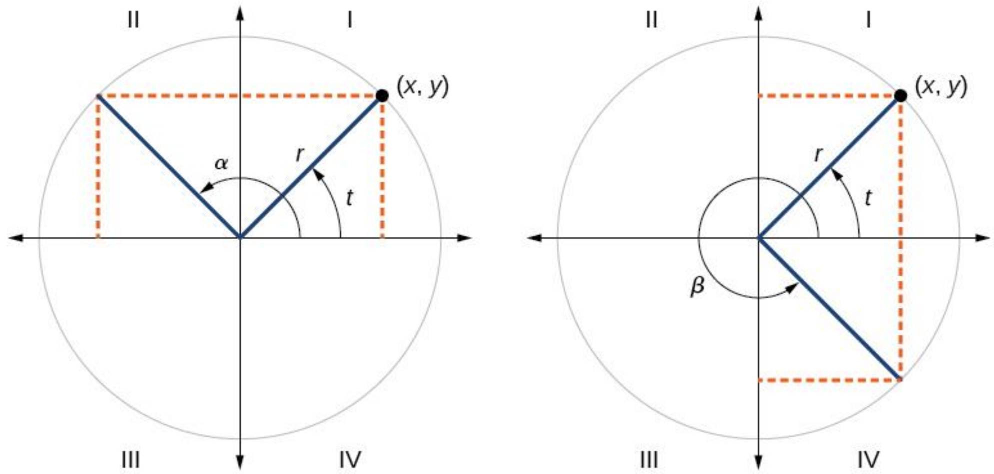
What are the domains of the sine and cosine functions? That is, what are the smallest and largest numbers that can be inputs of the functions? Because angles smaller than 0 and angles larger than 2π can still be graphed on the unit circle and have real values of *x*, *y*, and *r*, there is no lower or upper limit to the angles that can be inputs to the sine and cosine functions. The **input** to the sine and cosine functions is the rotation from the positive *x*-axis, and that may be ________ _____________ ______________________.

What are the ranges of the sine and cosine functions? What are the least and greatest possible values for their output? We can see the answers by examining the unit circle, as shown above. The bounds of the *x*-coordinate are [____, ____].The bounds of the *y*-coordinate are also[____,______].Therefore, the range of both the sine and cosine functions is[−1,1].

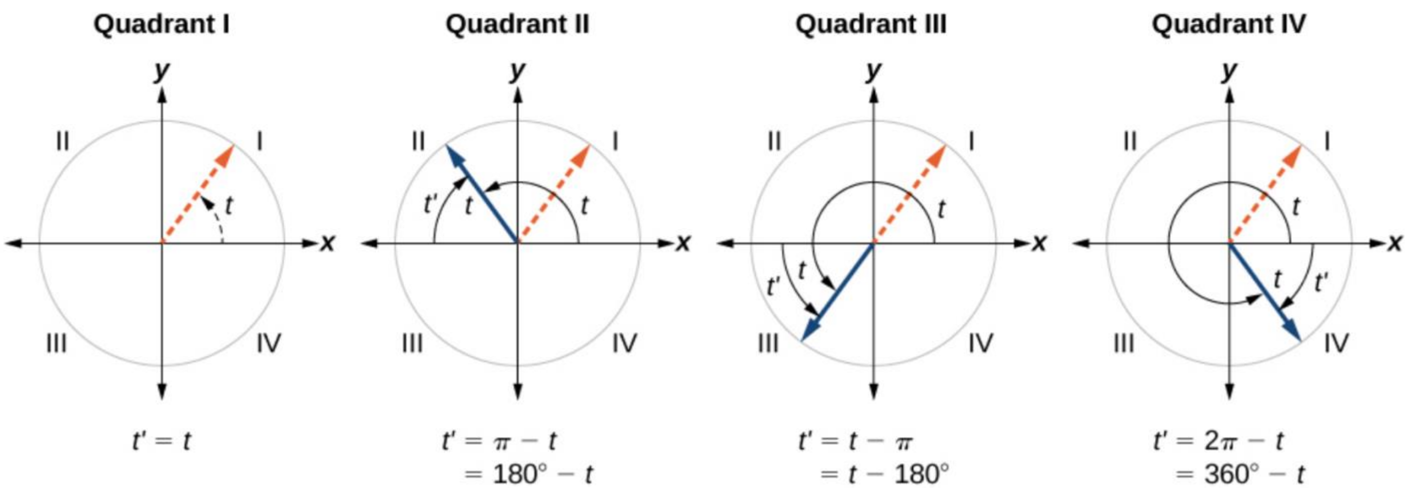
Finding Reference Angles

For any given angle in the first quadrant, there is an angle in the second quadrant with the same sine value. Because the sine value is the ___-coordinate on the unit circle, the other angle with the same ______ will share the same *y*-value, but have the _____________________ *x*-value. Therefore, its _______ value will be the opposite of the first angle's cosine value. Likewise, there will be an angle in the fourth quadrant with the _________ cosine as the original angle. The angle with the same cosine will share the _________ *x*-value but will have the opposite *y*-value. Therefore, its sine value will be the opposite of the original angle's sine value.

$$sin(t) = sin(\alpha) \text{ and } cos(t) = -cos(\alpha)$$
$$sin(t) = -sin(\beta) \text{ and } cos(t) = cos(\beta)$$



Recall that an angle's reference angle is the acute angle, *t*, formed by the terminal side of the angle *t* and the horizontal axis. A reference angle is always an angle between 0 and 90°,or 0 and $\frac{\pi}{2}$ radians.



How To:

Given an angle between 0 **and** 2π, **find its reference angle.**

1. An angle in the first quadrant is its own reference angle.
2. For an angle in the second or third quadrant, the reference angle is $|\pi - t|$ or $|180° - t|$.
3. For an angle in the fourth quadrant, the reference angle is $2\pi - t$ or $360° - t$.
4. If an angle is less than 0 or greater than 2π, add or subtract 2π as many times as needed to find an equivalent angle between 0 and 2π.

Example Finding a Reference Angle

Find the reference angle of 120°.

Find the reference angle of 225°

Try It: Find the reference angle of $\frac{5\pi}{3}$.

Using Reference Angles to Evaluate Trigonometric Functions

A GENERAL NOTE: USING REFERENCE ANGLES TO FIND COSINE AND SINE
Angles have cosines and sines with the same absolute value as their reference angles. The sign (positive or negative) can be determined from the quadrant of the angle.

How To:
Given an angle in standard position, find the reference angle, and the cosine and sine of the original angle.

1. Measure the angle between the terminal side of the given angle and the horizontal axis. That is the reference angle.
2. Determine the values of the cosine and sine of the reference angle.
3. Give the cosine the same sign as the x-values in the quadrant of the original angle.
4. Give the sine the same sign as the y-values in the quadrant of the original angle.

Example **Using Reference Angles to Find Sine and Cosine**
Using a reference angle, find the exact value of cos(150°) and sin(150°).

Using the reference angle, find cos $5\pi/4$ and sin $5\pi/4$.

Try It:

a. Use the reference angle of 315° to find cos (315°) and sin(315°).
b. Use the reference angle of $-\frac{\pi}{6}$ to find $\cos(-\frac{\pi}{6})$ and $\sin(-\frac{\pi}{6})$.

Example **Evaluate with Sine and Cosine of Special Angles**
Evaluate $\cos(7\pi/6)\sin(3\pi/2)$.

How To:

Given the angle of a point on a circle and the radius of the circle, find the (x, y) **coordinates of the point.**

1. Find the reference angle by measuring the smallest angle to the *x*-axis.
2. Find the cosine and sine of the reference angle.
3. Determine the appropriate signs for x and y in the given quadrant.

Example **Using the Unit Circle to Find Coordinates**

Find the coordinates of the point on the unit circle at an angle of $7\pi/6$.

Try It: Find the coordinates of the point on the unit circle at an angle of $\frac{5\pi}{3}$.

Find an angle θ with $0° < \theta < 360°$ that has the same : Sin(20°) and Cos(20°).

Find the coordinates of a point on circle with radius 30 corresponding to an angle of 225° .

If $t = -\frac{11\pi}{6}$, find the terminal point P(x,y) on the unit circle, x---------------, y-------------------.

Compute the exact value of each of the following:

Sin $(-\frac{15\pi}{4})$=-----------------------   Cos $(-\frac{15\pi}{4})$=-----------------------   Tan $(-\frac{15\pi}{4})$=-----------------------

Find the terminal point P(x,y) on the unit circle determined by the given value of t:

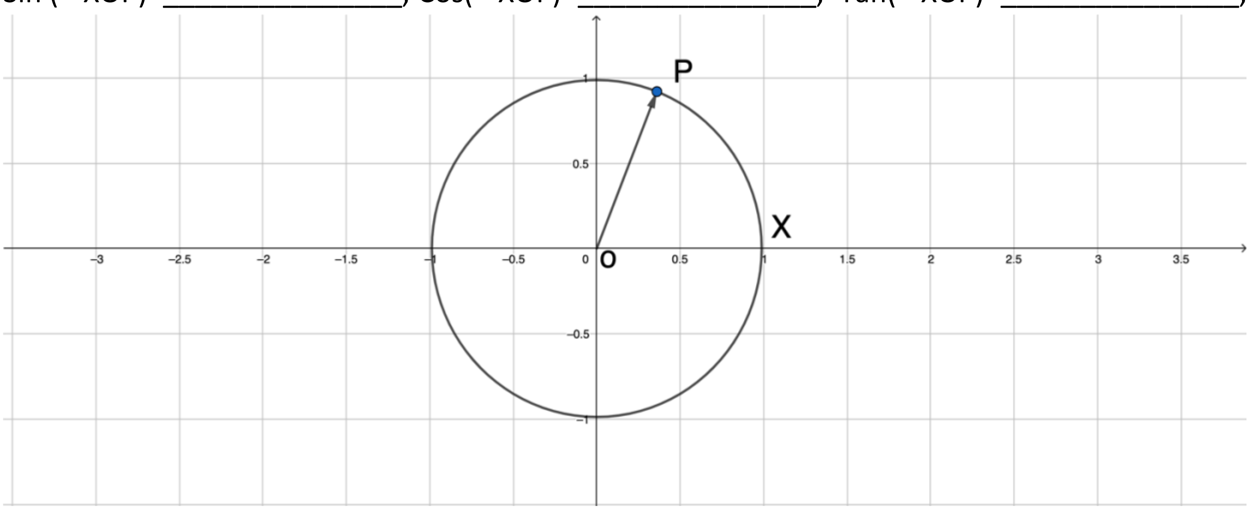
a. If t = $\frac{\pi}{2}$, then x =___________, y______________.
b. If t = -$\frac{\pi}{2}$, then x =___________, y______________.
c. If t = $\frac{\pi}{3}$, then x =___________, y______________.
d. If t = -$\frac{\pi}{3}$, then x =___________, y______________.
e. If t = $\frac{3\pi}{4}$, then x =___________, y______________.
f. If t = -$\frac{3\pi}{4}$, then x =___________, y______________.

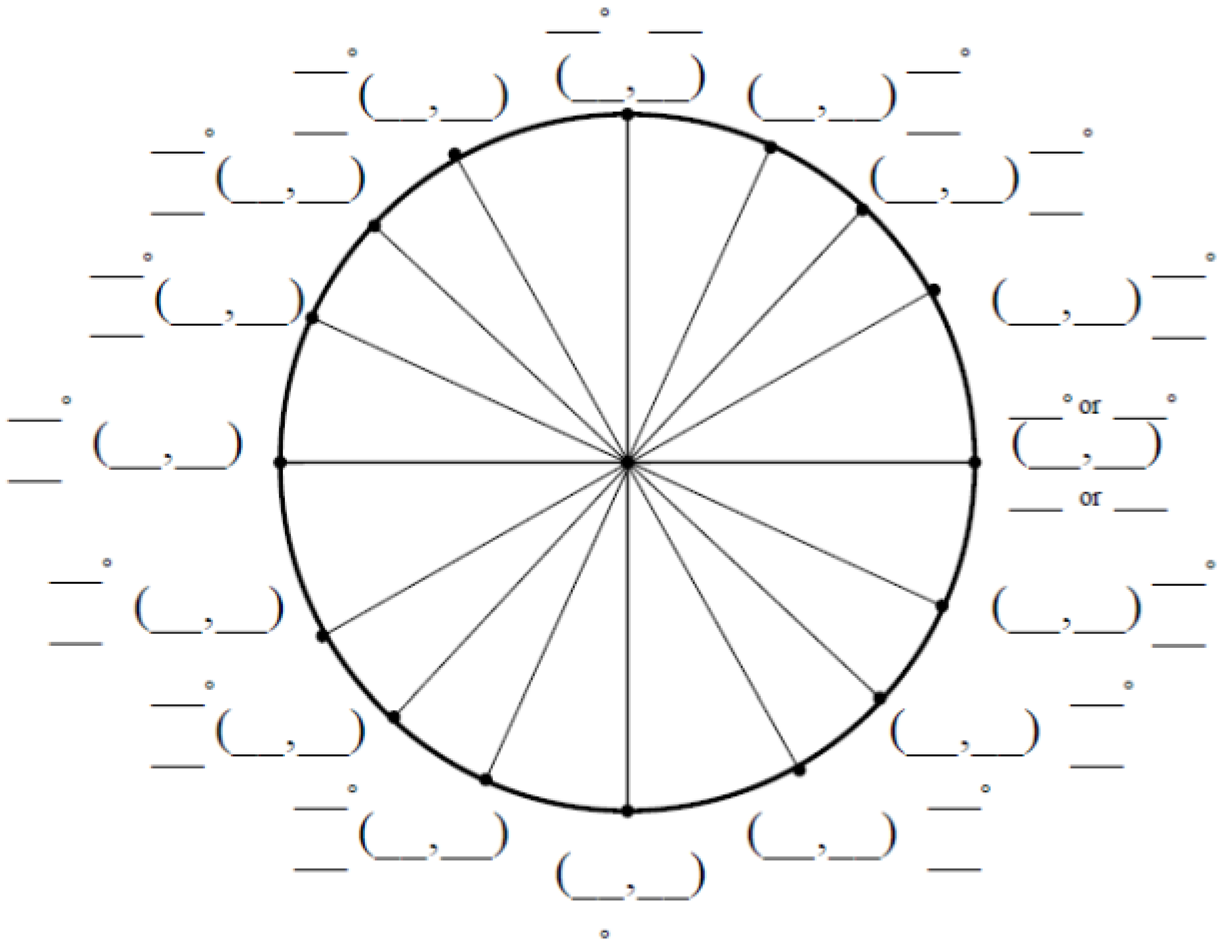
For each value of t below, determine the quadrant in which the terminal point is found and find the corresponding reference number t.

a. t= $\frac{2\pi}{3}$ is found in quadrant ______________, and t'__________.
b. t= $\frac{5\pi}{4}$ is found in quadrant ______________, and t'__________.
c. t= $\frac{19\pi}{6}$ is found in quadrant ______________, and t'__________.
d. t= 3 is found in quadrant ______________, and t'__________.

The graph shows a unit circle with point P at $(\frac{7}{25}, \frac{24}{25})$
Enter the exact values or DNE if the value is undefined.

Sin (<XOP)=_______________, Cos(<XOP)=_______________, Tan(<XOP)=_______________,

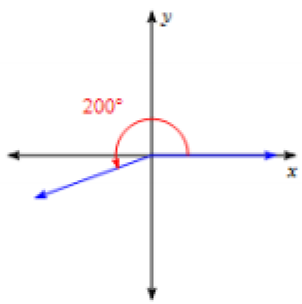
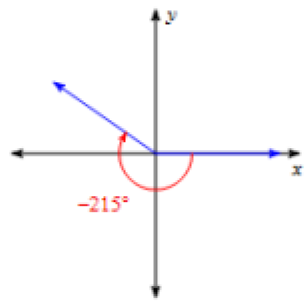
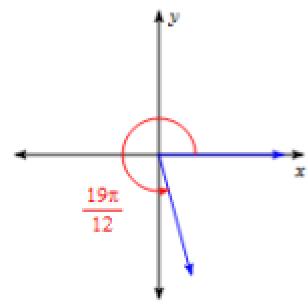
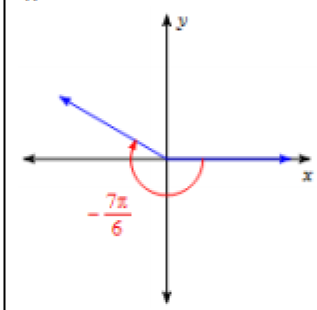




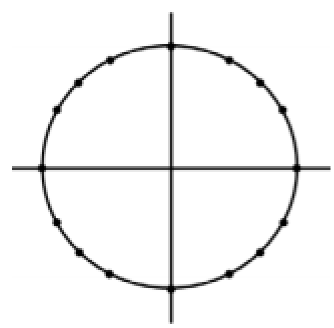
Fill in the missing parts of the table.

degrees	radians	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$	- degree	- radian
	$\frac{\pi}{3}$								
		$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$						
								$-120°$	
			-1						

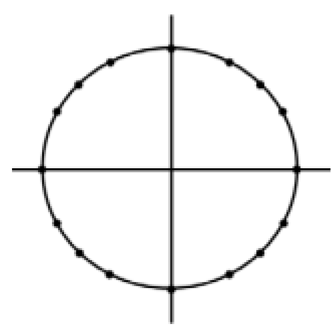
Find the reference angle.

1. 	2. 	3. 	4. 
5. $-130°$	6. $230°$	7. $-\frac{13\pi}{9}$	8. $\frac{3\pi}{4}$

Find the exact value.

9. $\sin 90° =$	10. $\cos 120° =$	11. $\tan 45° =$	
12. $\tan 120° =$	13. $\cos 225° =$	14. $\sin 135° =$	
15. $\sin 330° =$	16. $\tan 315° =$	17. $\cos 240° =$	
18. $\sin(-225°) =$	19. $\cos(-240°) =$	20. $\tan(-300°) =$	
21. $\sec(180°) =$	22. $\csc(-270°) =$	23. $\cot(-315°) =$	

Find the exact value.

24. $\sin\frac{\pi}{2} =$	25. $\tan\frac{\pi}{4} =$	26. $\cos\frac{3\pi}{2} =$	
27. $\cos\frac{4\pi}{3} =$	28. $\cos\frac{\pi}{6} =$	29. $\tan\pi =$	
30. $\sin\frac{5\pi}{4} =$	31. $\cos\frac{5\pi}{3} =$	32. $\sin\frac{5\pi}{6} =$	
33. $\tan\frac{7\pi}{4} =$	34. $\sin(-\pi) =$	35. $\tan\left(-\frac{3\pi}{2}\right) =$	
36. $\cos\left(-\frac{\pi}{3}\right) =$	37. $\sec\left(-\frac{\pi}{2}\right) =$	38. $\sin\left(-\frac{5\pi}{4}\right) =$	