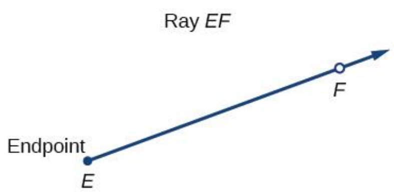
7.1 – Angles

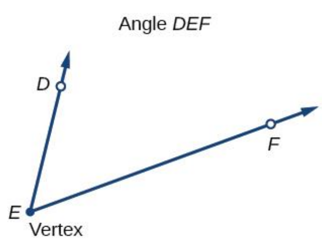
Learning Objectives
In this section you will:

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

Drawing Angles in Standard Position



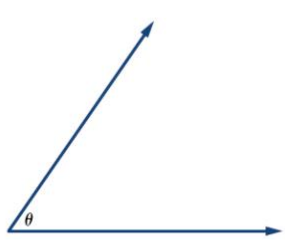
A **Ray** is a directed line segment. It consists of one point on a line and all points extending in one direction from that point. The first point is called the **Endpoint** of the ray.

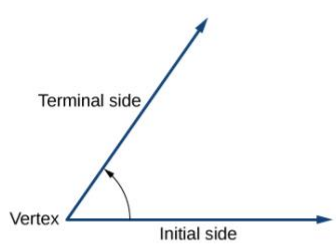


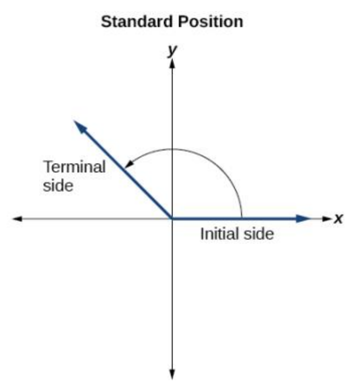
An **Angle** is the union of two rays having a common endpoint. The endpoint is called the **Vertex** of the angle, and the two rays are the sides of the angle.

Greek letter are often used as variables for the measure of an angle.

θ	theta	$\phi or \varphi$	phi	α	alpha
β	beta	γ	gamma		





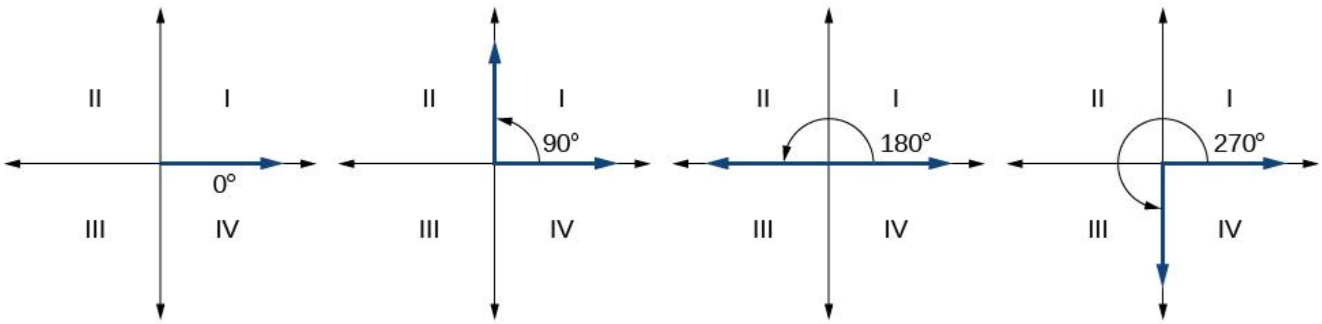


The measure of an angle is the amount of rotation from the **initial** side to the **terminal** side. Probably the most familiar unit of angle measurement is the **Degree .** One degree is $\frac{1}{360}$ of a circular rotation, so a complete circular rotation contains 360 degrees. An angle measured in degrees should always include the unit “degrees” after the number, or include the degree symbol **°.** For example, 90 degrees = 90°.

An angle is in **Standard p**osition if its vertex is located at the origin, and its initial side extends along the positive *x*-axis.

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be **a positive** angle. If the angle is measured in a clockwise direction, the angle is said to be a **negative** angle.

A GENERAL NOTE: QUADRANTAL ANGLES
An angle is a **quadrantal angle** if its terminal side lies on an axis, including 0°, 90°, 180°, 270°, or 360°.



Drawing Angles in Standard Position
Example: Drawing an Angle in Standard Position Measured in Degrees
Sketch an angle of −120° in standard position.

Try It: Show an angle of 240° on a circle in standard position.

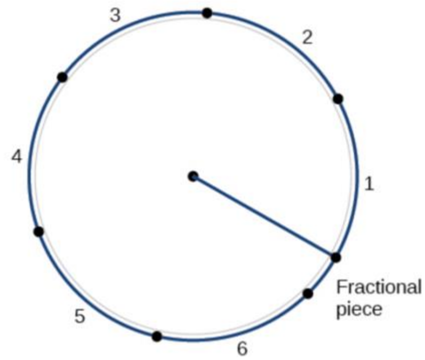
Converting Between Degrees and Radians
An ______________ may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The ________________ of the arc around an entire circle is called the ____ of that circle.

The Circumference of a Circle

$$\boldsymbol{C = 2\pi r}$$

$$\boldsymbol{\frac{C}{r} = 2\pi}$$

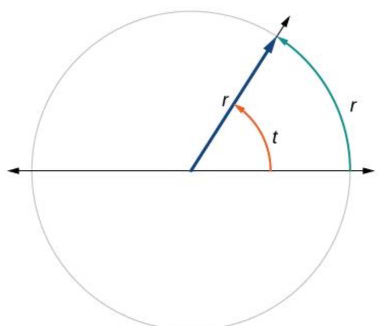
$$2\pi \approx 6.28$$



That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh.

What Is a Radian?

One ____________________________ is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals _______ times the ___________, a full circular rotation is 2π radians.



1 rotation = $360° = 2\pi$ radians

$\frac{1}{2}$ rotation = $180° = \pi$ radians

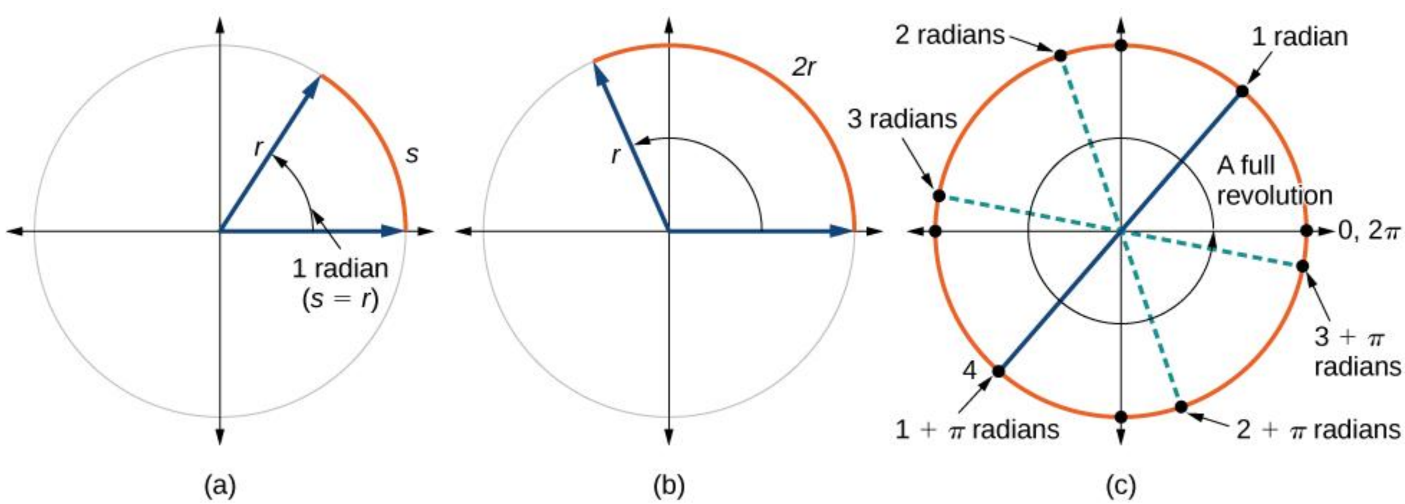
$\frac{1}{4}$ rotation = $90° = \frac{\pi}{2}$ radians

Relating Arc Lengths to Radius

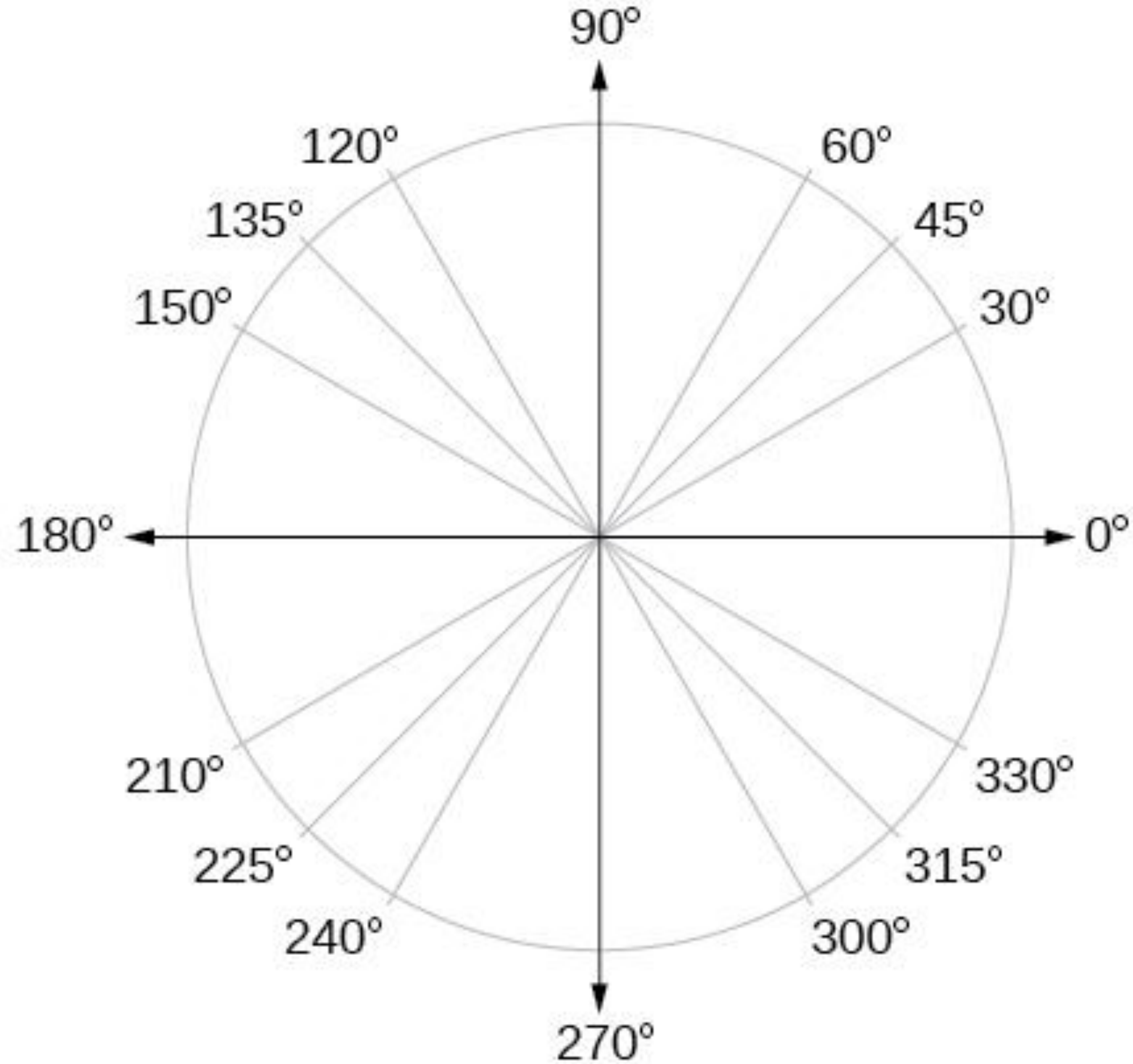
An ___________ ___________________, *s,* is the length of the curve along the arc of a circle.

$$s = r\theta \qquad\qquad \theta = \frac{s}{r}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$radian.



Identifying Special Angles Measured in Radians



Example Finding a Radian Measure

Find the radian measure of one-third of a full rotation.

Try It: Find the radian measure of three-fourths of a full rotation.

Converting Between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion where

θ is the measure of the angle in degrees and

θ_R is the measure of the angle in radians.

A GENERAL NOTE: CONVERTING BETWEEN RADIANS AND DEGREES

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{\theta_R}{\pi}$$

Examples:

Convert each radian measure to degrees.

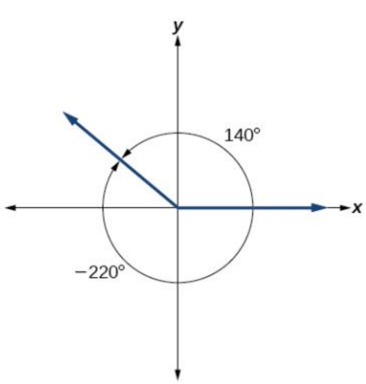
1. $\frac{\pi}{6}$
2. 3
3. $-\frac{3\pi}{4}$
4. $\frac{5\pi}{6}$

Convert each degrees to radian measure.

1. 15 deg
2. 126 deg
3. -75 deg
4. 126 deg

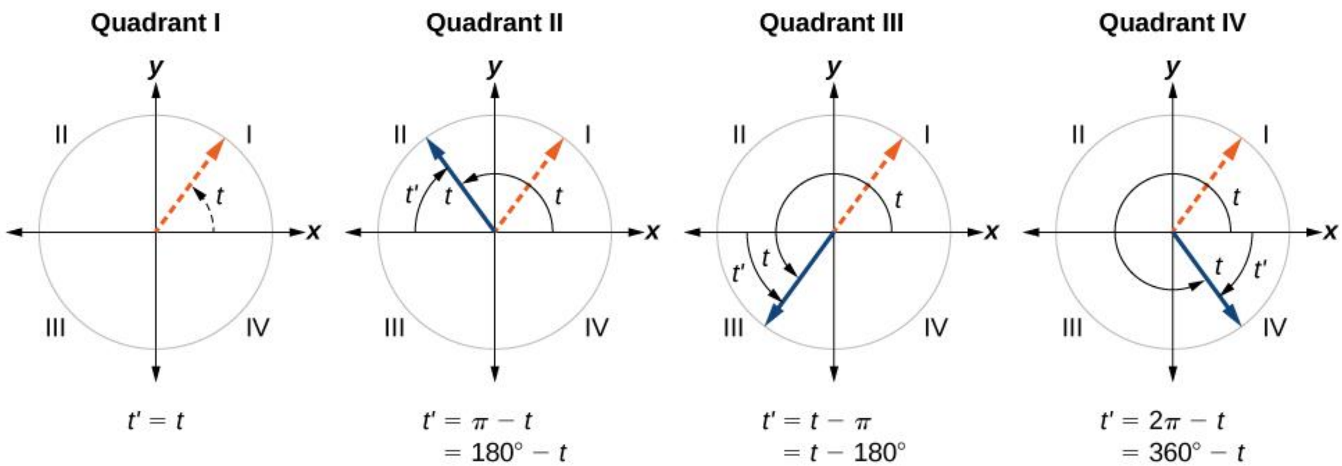
Coterminal Angles

If two angles in standard position have the same terminal side, they are ___________________________ angles.



Any angle has infinitely many coterminal angles because each time we add 360° to that angle - or subtract 360° from it - the resulting value has a terminal side in the same location. For example 100 deg and 460 deg are coterminal for this reason, as is -260 deg

An angle's ______________________________ angle is the measure of the __________________, ______________________, acute angle t formed by the terminal side of the angle t and the _________________________ axis.



How To:

Given an angle greater than360°, find a coterminal angle between 0° and 360°

1. Subtract 360° from the given angle.
2. If the result is still greater than 360°, subtract 360° again till the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Given an angle with measure less than 0, **find a coterminal angle having a measure between** 0° **and** 360°.

1. Add 360° to the given angle.
2. If the result is still less than0°, add 360° again until the result is between 0° and 360°.
3. The resulting angle is coterminal with the original angle.

Example Finding an Angle Coterminal with an Angle of Measure Greater Than 360°

Find the least positive angle ϑ that is coterminal with an angle measuring 420°, where $0° \le \vartheta < 360°$.

Try It: Find an angle α that is coterminal with an angle measuring 870°, where $0° \le \alpha < 360°$.

Example Finding an Angle Coterminal with an Angle Measuring Less Than 0°

Show the angle with measure −450° on a circle and find a positive coterminal angle α such that $0° \le \alpha < 360°$.

Try It: Find an angle β that is coterminal with an angle measuring −300° such that $0° \le \beta < 360°$.

Try it: Write an expression describing all the angles that are coterminal with 194°. Please use the variable k in you answer. Give your answer in degress.

Try it: Are the standard position angles measuring 116° and 476° coterminal?

Finding Coterminal Angles Measured in Radians

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Example Finding Coterminal Angles Using Radians

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \le \beta < 2\pi$.

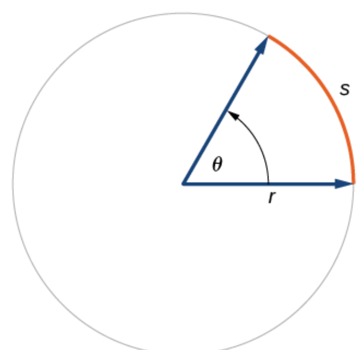
Try It: Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17}{6}\pi$ where $0 \le \theta < 2\pi$.

Determining the Length of an Arc

GENERAL NOTE: ARC LENGTH ON A CIRCLE

In a circle of radius *r*, the length of an arc s subtended by an angle with measure θ in radians, shown in Figure, is

$s = r\theta$



How To:

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r θ : $s = r\theta$.

Example Finding the Length of an Arc

Find the length of the arc of a circle of diameter 14 meters subtended by the central angle of $\frac{5\pi}{6}$ radians.

Try It: Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Example Finding the Length of an Arc
Find the distance along an arc on the surface of Earth that subtends a central angle of 5°. The radius of Earth is 3,960 mi.

Example **Finding the Length of an Arc**
Find the distance that the earth travels in one day in its path around the sun. Assume that a year has 365 days and the path of the earth around the sun is a circle of radius of radius of 93 million miles. Round your answer to one decimal place, which is the hundredth thousands.

1 rotation corresponds to $2\pi r$ complete circumference of a circle.

That is 1 rotation corresponds to $2\pi * 93 = 186\pi$ million miles.

Now, 1 rotation takes place in 365 days.

That means, in 365 days, it travels 186π million miles.

So in 1 days, it should travel less miles, so divide : $\frac{186\pi}{365}$=1.6 million miles

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

a. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?

Note that Angular velocity is directly related to rotations per time. If you complete 1 rotation, you make 360 degree. We use this fact here.

1 rotation corresponds to $2\pi r$ complete circumference of a circle.

That is 1 rotation corresponds to $2\pi * 93 = 186\pi$ million miles.

Now, 1 rotation takes place in 365 days.

That means, in 365 days, it travels 186π million miles.

So in 1 days, it should travel less miles, so divide : $\frac{186\pi}{365}$=1.6 million miles

b. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Note that Angular velocity is directly related to rotations per time. If you complete 1 rotation, you make 360 degree. We use this fact here.

Finding the Area of a Sector
In addition to arc length, we can also use angles to find the area of a ___________________ of a circle. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie.

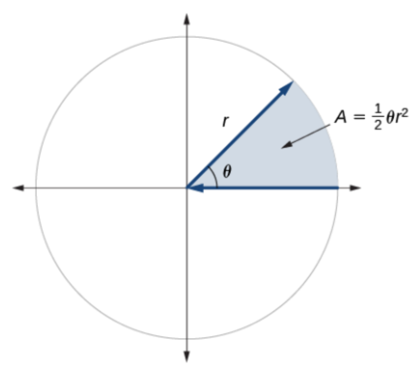
AREA OF A SECTOR
The area of a sector of a circle with radius r subtended by an angle θ, measured in radians, is
$A = \frac{1}{2}\theta r^2$

Note: Area of sector

$$= \left(\frac{\theta}{2\pi}\right)\pi r^2$$

$$= \frac{\theta\pi r^2}{2\pi}\pi r^2$$

$$= \frac{1}{2}\theta r^2$$



How To:
Given a circle of radius r, find the area of a sector defined by a given angle θ.
1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r :

$$A = \frac{1}{2}\theta r^2.$$

Example Finding the Area of a Sector
A sector of a circle with diameter 10 feet and an angle of 90°. Find the area of the sector.

Use Linear and Angular Speed to Describe Motion on a Circular Path
An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its displacement) in a given time interval.

$$v = \frac{s}{t}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval.

$$\omega = \frac{\theta}{t}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v = r\,\omega$$

Given the amount of angle rotation and the time elapsed, calculate the angular speed.

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega = \frac{\theta}{t}$
3. The resulting speed will be in radians per time unit.

A hamster runs at a speed of 14 centimeters per second in a wheel of radius 8 centimeters.

a. **What is the angular velocity of a wheel ? (in radians/sec)__________________**

Ans: Learn the difference between angular velocity and linear velocity. Angular velocity is measured in terms of rotations per minute seconds, hours) or radians per minutes(seconds, hours). Angular velocity is change in angle with respect to time. However, linear velocity is change in distance (meters, miles, Centimeters so on) with respect to time. By Distance, we mean distance traveled in circumference when we talk about circular motion.

It is linear speed that is given in the question, (centimeters per second) . So v=14 cm. Now we use the formula, v= wr, where w is angular velocity and r is radius.

$14 = w8$

$\frac{14}{8} = \frac{7}{4} = w$ (Dividing both sides by 8

So, $w = \frac{7}{4}$ radians/sec

b. **What fast will the wheel spin in revolutions per min? ______________________**

Note that Angular velocity is directly related to rotations per time. If you complete 1 rotation, you make 360 degree. We use this fact here.

$2\pi\ radians$ corresponds to 1 revolutions.

$\frac{7}{4}$ radians corresponds to $2\pi * \frac{7}{4} = \frac{7}{2}\pi$ revolutions.

But, note that this answer is in terms of seconds, so we convert it in terms of minutes.

$$\frac{7}{2}\pi\ \frac{revolutions}{seconds} * \frac{60seconds}{1\ min} = 210\pi\ revolutions/min$$

How To:

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed.

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega = \frac{\theta}{t}$
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply v = rω.

Example **Finding a Linear Speed**

A hamster running in a wheel of radius of 9cm spins the wheel one revolution in 3 seconds.

a. What is the angular velocity of the wheel ? (in radians/sec)?

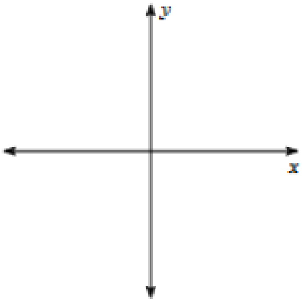
b. At what linear velocity is the hamster running? (in cm/sec)?

Try It: A satellite is rotating around Earth at 0.25 radian per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

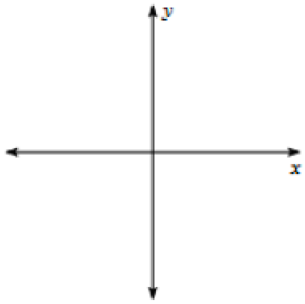
Practice

Draw an angle with the given measure in standard position.

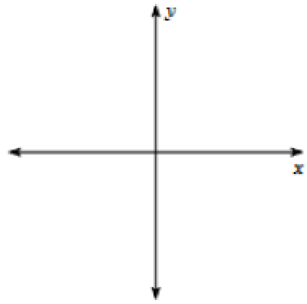
1. 330°



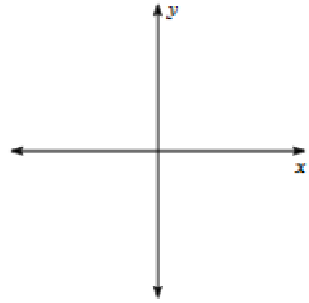
2. −115°



3. −290°

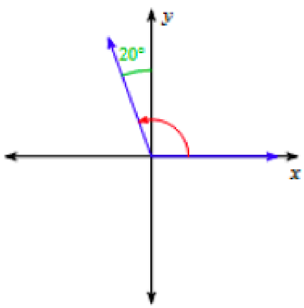


4. 440°



Find the measure of each angle.

5.



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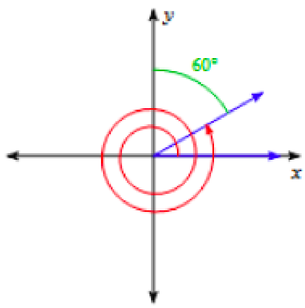

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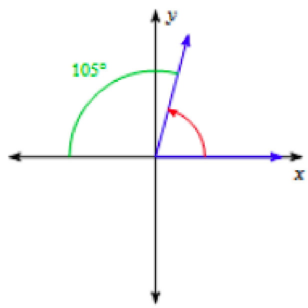

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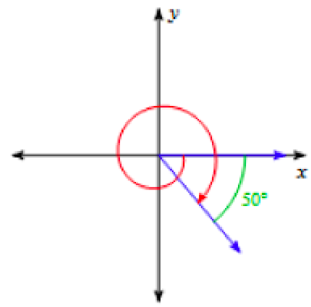


State the quadrant in which the terminal side of each angle lies.

9. −446°	10. 870°	11. −190°	12. 215°

Find one positive and one negative coterminal angle the angle given.

13. 30°	14. −705°

Find a coterminal angle between 0° and 360°.

15. −45°	16. 435°

1. Mr. Bean loves the half pipe and wants to try a Ollie Fakey-to-Air Double McTwist 1280.

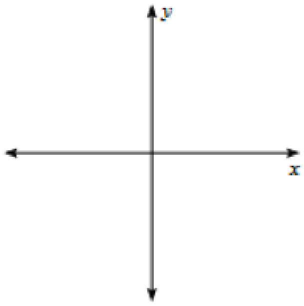
   a. How many revolutions will he make to complete the 1280?

   b. If Bean leaves the half pipe at the exact same spot, is a Poptart 540 coterminal to a Lando-Roll 1440? Justify.

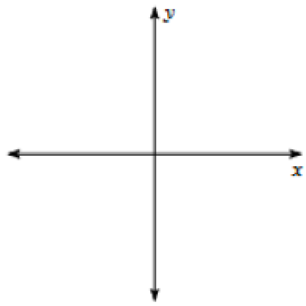
2. Mr. Brust rides his unicycle over a line of fresh paint. He continues to ride in a straight line leaving marks that are 6.5 feet apart. What is the radius of Mr. Brust's unicycle tire?

Draw an angle with the given measure in standard position.

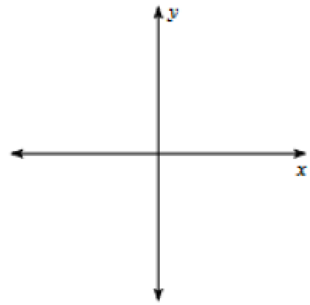
1. $\frac{\pi}{6}$



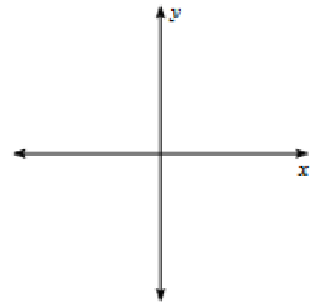
2. $-\frac{5\pi}{3}$



3. $\frac{3\pi}{4}$

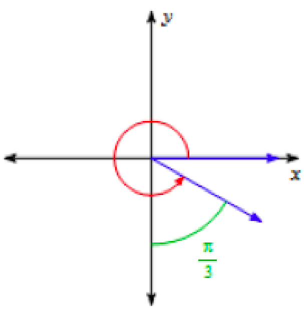


4. $-\frac{12\pi}{5}$



Find the measure of each angle. (IN RADIANS!)

5.



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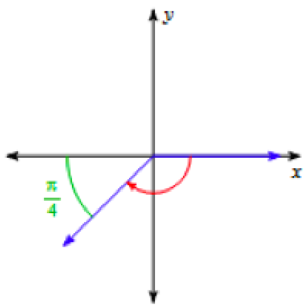

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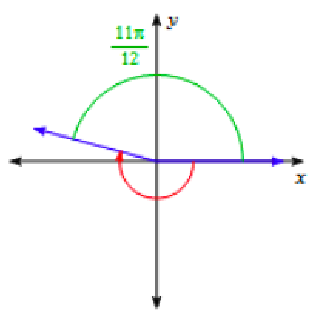

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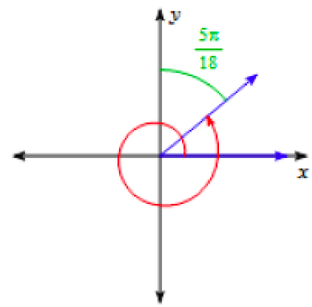


State the quadrant in which the terminal side of each angle lies.

9. $\frac{15\pi}{4}$

10. $\frac{5\pi}{6}$

11. $-\frac{10\pi}{9}$

12. $-\frac{17\pi}{6}$

Find one positive and one negative coterminal angle the angle given. (IN RADIANS!)

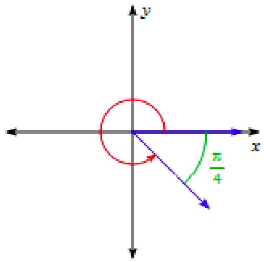
13. $\frac{\pi}{3}$

14. $\frac{5\pi}{4}$

Find a coterminal angle between 0 and 2π.

15. $\frac{9\pi}{4}$

16. $\frac{13\pi}{2}$

Find ALL coterminal angles in the world for each angle.	
17. 	18. $\frac{\pi}{2}$

Convert each degree measure into radians.	
19. $225°$	20. $280°$
21. $-210°$	22. $-1020°$

Convert each radian measure into degrees.	
23. $-\frac{5\pi}{9}$	24. $\frac{5\pi}{6}$
25. $\frac{23\pi}{36}$	26. $\frac{79\pi}{18}$

1. **A circular blade with a 12-inch diameter spins at a rate of 1800 rpm (revolutions per minute).**

    a. What is the blade's angular velocity in radians per minute?

    b. Find the linear velocity (in inches per minute) of one of the teeth on the edge of the blade.

    c. Convert the linear velocity into feet per second.

2. **Vinyl record albums are 11 inches in diameter and spin at a rate of 33 rpm.**

    a. What is a record's angular velocity in radians per minute?

    b. How fast (in inches per minute) would a record move under a needle placed on the records edge?

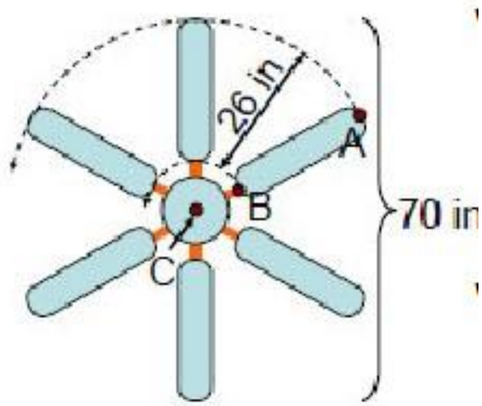
    c. Convert this linear velocity to feet per second.

3. **The blades of a ceiling fan are 26 inches long, but the fan's entire diameter in 70 inches. It spins at a rate of 100 rpm.**

    a. What is the linear velocity of a point on the outer edge of the blade?

    b. What is the linear velocity of a point on the inner edge of the blade?

70 in

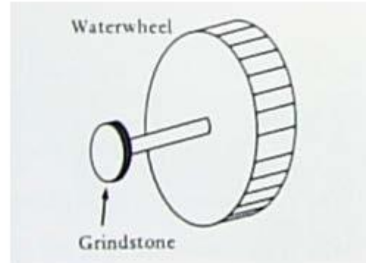


    c. What is the linear velocity of a point at the center of the fan?

    d. Do point A and point B have the same angular velocity OR the same linear velocity?

4. A waterwheel of diameter 12 feet turns at .3 radians per second.

a. What is the linear velocity of the rim?

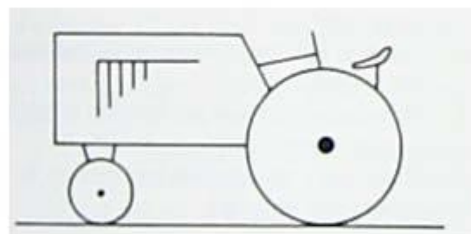


b. The wheel is connected by an axel to a grindstone of diameter 3 feet.
What is the linear velocity of a point on the rim of the grindstone?

c. Do a point on the waterwheel and a point on the grindstone have the same angular velocity OR the same linear velocity?

5. The rear wheels of a tractor are 4 feet in diameter and turn at 20 rpm.

a. How fast is the tractor going in feet per second?



b. The front wheels have a diameter of only 1.8 feet.
What is the linear velocity of a point on their tire treads?

c. What is the angular velocity of the front wheels in rpm?

c. Do a point on the rear wheel and a point on the front wheel have the same angular velocity OR the same linear velocity?

Example **Finding the Length of an Arc**

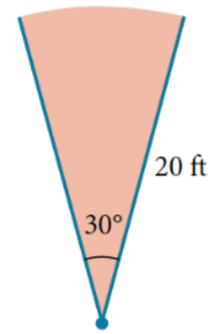
Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

a. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?

b. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

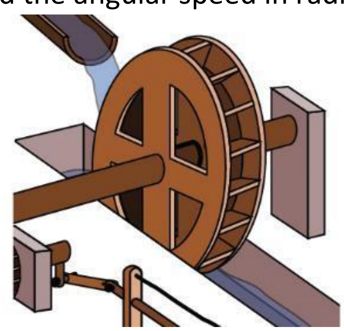
Example Finding the Area of a Sector

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in Figure 23. What is the area of the sector of grass the sprinkler waters?



Try It: In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

A water wheel, shown in **Figure 24**, completes 1 rotation every 5 seconds. Find the angular speed in radians per second.



Try It: An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road