1 Algebra

1.1 Exponential Properties

(i) $x^0 = 1$
(ii) $x^n x^m = x^{n+m}$
(iii) $\frac{x^n}{x^m} = x^{n-m} = \frac{1}{x^{m-n}}$
(iv) $(x^n)^m = x^{nm}$
(v) $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$
(vi) $x^{-n} = \frac{1}{x^n}$
(vii) $\frac{1}{x^{-n}} = x^n$
(viii) $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n = \frac{y^n}{x^n}$
(ix) $x^{\frac{n}{m}} = \left(x^{\frac{1}{m}}\right)^n = (x^n)^{\frac{1}{m}} = \sqrt[m]{x^n}$

1.2 Logarithm Properties

(i) $\log_n(0) = \mathit{Undefined}$
(ii) $\log_n(1) = 0$
(iii) $\log_n(n) = 1$
(iv) $\log_n(n^x) = x$
(v) $n^{\log_n(x)} = x$
(vi) $\log_n(x^r) = r\log_n(x) \neq \log_n^r(x) = (\log_n(x))^r$
(vii) $\log_n(xy) = \log_n(x) + \log_n(y)$
(viii) $\log_n\left(\frac{x}{y}\right) = \log_n(x) - \log_n(y)$
(ix) $-\log_n(x) = \log_n\left(\frac{1}{x}\right)$
(x) $\frac{\log(x)}{\log(n)} = \log_n(x)$

1.3 Radical Properties

(i) $\sqrt[n]{x} = x^{\frac{1}{n}}$
(ii) $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$
(iii) $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$
(iv) $\sqrt[n]{\frac{x}{y}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}}$
(v) $\sqrt[n]{x^n} = x, \quad \text{if } n \text{ is odd}$
(vi) $\sqrt[n]{x^n} = |x|, \quad \text{if } n \text{ is even}$

1.4 Absolute Value Properties

(i) $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
(ii) $|x| \geq 0$
(iii) $|-x| = |x|$
(iv) $|ca| = c|a|, \quad \text{if } c > 0$
(v) $|xy| = |x||y|$
(vi) $|x^2| = x^2$
(vii) $|x^n| = |x|^n$
(viii) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$
(ix) $|a-b| = b-a$, if $a \leq b$
(x) $|a+b| \leq |a| + |b|$
(xi) $|a| - |b| \leq |a-b|$

1.5 Factorization

(i) $x^2 - a^2 = (x+a)(x-a)$
(ii) $x^2 + 2ax + a^2 = (x+a)^2$
(iii) $x^2 - 2ax + a^2 = (x-a)^2$
(iv) $x^2 + (a+b)x + ab = (x+a)(x+b)$
(v) $x^3 + 3ax^2 + 3a^2x + a^3 = (x+a)^3$
(vi) $x^3 - 3ax^2 + 3a^2x - a^3 = (x-a)^3$
(vii) $x^3 + a^3 = (x+a)(x^2 - ax + a^2)$
(viii) $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$
(ix) $x^{2n} - a^{2n} = (x^n - a^n)(x^n + a^n)$

1.6 Complete The Square

$$\boldsymbol{ax^2 + bx + c = 0 \quad \Rightarrow \quad a(x+d)^2 + e = 0}$$

- $d = \frac{b}{2a}$
- $e = c - \frac{b^2}{4a}$

1.7 Quadratic Formula

$$\boldsymbol{ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

- *If* $b^2 - 4ac > 0 \Rightarrow$ *Two real unequal solutions.*
- *If* $b^2 - 4ac = 0 \Rightarrow$ *Two repeated real solutions.*
- *If* $b^2 - 4ac < 0 \Rightarrow$ *Two complex solutions.*

2 Functions

2.1 Domain

- **Fractions** denominator $\neq 0$.
- **Logarithms** if the base is a number, the argument must be > 0, if the base depends on a variable, the base must be $> 0 \wedge \neq 1$.
- **Roots** with even index, the argument must be ≥ 0, for roots with odd index the domain is $\mathbb{R}$.
- **Arccos/Arcsin** the agrument must be $\in [-1, 1]$. For other trig functions we use trig properties to change them to cos and sin.
- **Exponential** base > 0.

2.2 Parity

We consider the partiy of the function only if $Dom(f)$ is mirrored on the origin: $(Dom(f) = [-2, 2] \vee (-\infty, \infty) \vee (-\infty, -1] \cup [1, \infty])$.

- **Even function** (with respect to the y axis) if: $f(-x) = f(x)$.
- **Odd function** (with respect to the origin) if: $f(-x) = -f(x)$.
- In every other case the function is neither even nor odd.

2.3 Axis Intercept

- **X intercept** can be many; is calculated by solving $f(x) = 0$. If $f(x) = \frac{g(x)}{h(x)}$ we solve just $g(x) = 0$. The points are then $(x_i, 0)$.
- **Y intercept** can be just one; is calculated by setting $x = 0$, the point is then $(0, f(0))$. If $x = 0 \notin Dom(f)$ there is no Y intercept.

2.4 Sign

The sign can only change when there is a x intercept (if the function is continous), thus if we solve $f(x) \geq 0$ we get both the X intercepts and where the function is positive.

2.5 Asymptotes/Holes

- **Hole** at point $(x_0, f_{semplified}(x_0))$ if plugging the critical point x_0 in the numerator of f gives $\frac{0}{0}$.
- **Vertical** asymptote at a critical point x_0 if: $\lim_{x \to x_0^-} f(x) = \pm\infty$ (left at $x = x_0$) $\lim_{x \to x_0^+} f(x) = \pm\infty$ (right at $x = x_0$).
- **Horizontal** aysmptote (if domain is unlimited at $\pm\infty$) if: $\lim_{x \to +\infty} f(x) = k$ (right $y = k$) $\lim_{x \to -\infty} f(x) = h$ (left $y = h$).
- **Oblique** aysmptote (if domain is unlimited at $\pm\infty$) if: $\lim_{x \to +\infty} \frac{f(x)}{x} = m \wedge \lim_{x \to +\infty}[f(x) - mx] = q$ (right at $y = mx + q$) $\lim_{x \to -\infty} \frac{f(x)}{x} = m \wedge \lim_{x \to -\infty}[f(x) - mx] = q$ (left at $y = mx + q$).

2.6 Monotonicity

A funciton f is:

- **Monotonically increasing** if: $\forall x, y : x \leq y \Rightarrow f(x) \leq f(y)$
- **Monotonically decreasing** if: $\forall x, y : x \leq y \Rightarrow f(x) \geq f(y)$
- **Strictly increasing** if: $\forall x, y : x < y \Rightarrow f(x) < f(y)$
- **Strictly decreasing** if: $\forall x, y : x < y \Rightarrow f(x) > f(y)$

2.7 Max, Min

Calculate $f'(x) = 0$, then all the solutions x_i are our candidates, where for a small $\epsilon > 0$:

- **Max** if: $f'(x_i - \epsilon) > 0 \wedge f'(x_i + \epsilon) < 0$.
- **Min** if: $f'(x_i - \epsilon) < 0 \wedge f'(x_i + \epsilon) > 0$.
- **Inflection** if (use sign table): $f'(x_i - \epsilon) < 0 \wedge f'(x_i + \epsilon) < 0$, or $f'(x_i - \epsilon) > 0 \wedge f'(x_i + \epsilon) > 0$

If $f'(x) > 0$, then f is strictly increasing.
If $f'(x) < 0$, then f is strictly decreasing.
If $f'(x) = 0$ f is constant.

2.8 Convexity

- **Convex** ($\cup$) if: $f''(x) > 0$
- **Concave** ($\cap$) if: $f''(x) < 0$

2.9 Inflection Points

Calculate $f''(x) = 0$, then all the solutions x_i are our candidates (except where $f(x)$ is not defined), where for a small $\epsilon > 0$:

- **Increasing Inflection** if: $f''(x_i - \epsilon) < 0 \wedge f''(x_i + \epsilon) > 0$
- **Decreasing Inflection** if: $f''(x_i - \epsilon) > 0 \wedge f''(x_i + \epsilon) < 0$
- Otherwise nothing happens on x_i.