River Parishes Community College

Math 1100: College Algebra

Polynomials and Rational Functions

Chapter 5.2-5.3 Polynomials, Power Functions and their Graphs

Semester
Fall/Sp Year

Department
Physical Science: Math

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Learning Objectives

In this section, you will:
♣ Identify power functions.
♣ Identify end behavior of power functions.
♣ Identify polynomial functions.
♣ Identify the degree and leading coefficient of polynomial functions.

1 Introduction: Why Study Polynomials?

Imagine you are in first grade where you just learned only Natural numbers. Now you are asked to solve the following problems:

$$x + 2 = 0.$$
$$2x - 3 = 0$$
$$x^2 - 2 = 0$$
$$x^2 + 1 = 0$$

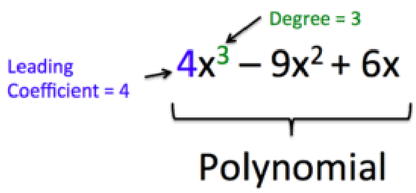
Then you can not solve any of these problems because you have not learned about negative, rational, irrational and complex numbers yet. The solution of the first equation is a negative number, the second one is a rational number, the third one is an irrational number and forth solution is a complex numbers. So one very important use of polynomials is to construct different kind of numbers.
For example: **The multiplication of 307 and 43 can be written as $(3x^2+7)(4x+3) = 12x^3+9x^2+28x+21$. If you now substitute 10 for x wherever it appears, you will find that you have one the original multiplication of the two numbers by polynomials. Thus our number system is a shorthand notation for the algebra of polynomials in one variable with variable replaced by the number** 10.
There are other reasons to study polynomials. Many mathematical processes that are done in everyday life can be interpreted as polynomials. Summing the cost of items on a grocery bill can be interpreted as a polynomial. Calculating the distance traveled of a vehicle or object can be interpreted as a polynomial. Calculating perimeter, area, and volume of geometric figures can be interpreted as polynomials. These are just some of the many applications of polynomials.

2 Polynomial Funcitons

1. **Monomials**: These are the objects with one term.
   - Example: $5x^3, 7x, 6, 9x^2$ etc.
2. **Binomials**: These are two terms objects.
   - Example :$5x^2 + 5x, 2x^9 - 5$
3. **Trionomials**: These are objects with three terms.
   - Example: $\frac{1}{2}x^2 + 2x + 3, 3x^3 - 2x^2 + x$.
4. **Polynomials** : one or more numbers of terms : Poly (numbers) + nomials (terms).
   - Example: $2, 2x, x^2 - 2x + 2, 5x^2 - \frac{3}{2}x + 9$.
5. **Examples that are Not Polynomials:**
   - $5x^{-2}$ (negative exponent).

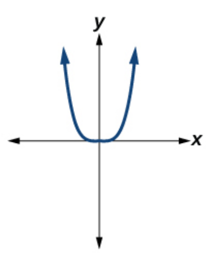
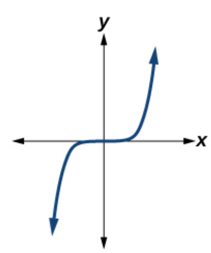
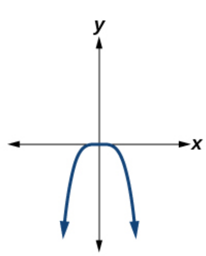
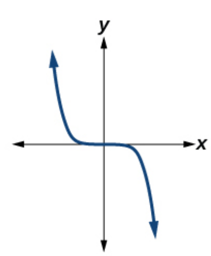
- $\frac{1}{x}$ (x on the denominator).
- $\frac{3x+3}{2x-6}$ (x on the denominator).
- $\sqrt{x+2}$ (square root).

A **polynomial function** of **degree** n is a function of the form $P(x) = a_n x^n + a_{n-1} x^{x-1} + ...+ a_1 x + a_0$, where n is a nonnegative integer and $a_n \neq 0$. The highest power (highest exponent) is called **degree** of that Polynomial. The numbers $a_0, a_1, a_2, ...a_n$ are called the **coefficients** of the polynomial. The number a_0 is the **constant coefficient** or constant term. The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term.**



Note: The degree of any non zero constant is zero. However, the degree of zero function i.e $f(x) = 0$ is not defined. why?
Graphs of polynomial functions are smooth curves with no breaks or corners. The end behavior of a polynomial is a description of what happens as x becomes large in the positive or negative direction.For any polynomial, the end behavior is determined by the term that contains the highest (degree) power of x.

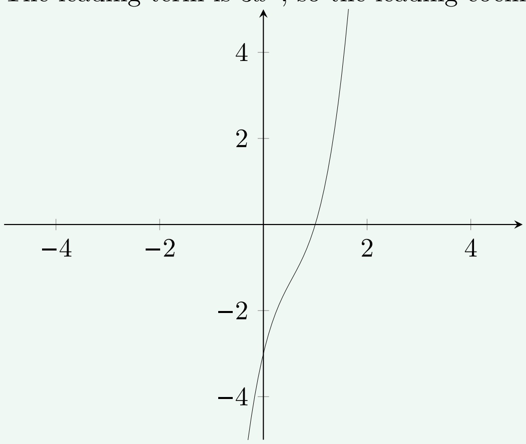
Notation:
$x \to \infty$ means "x becomes large in the positive direction"
$x \to -\infty$ means "x becomes large in the negative direction"

	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to \infty$	 $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to \infty$
Negative constant $k < 0$	 $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to -\infty$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to -\infty$

Example

Find the degree and leading coefficient of $f(x) = 3x^3 - 5x^2 + 5x - 3$.. Describe the behavior of $f(x)$ as $x \to \pm\infty$.

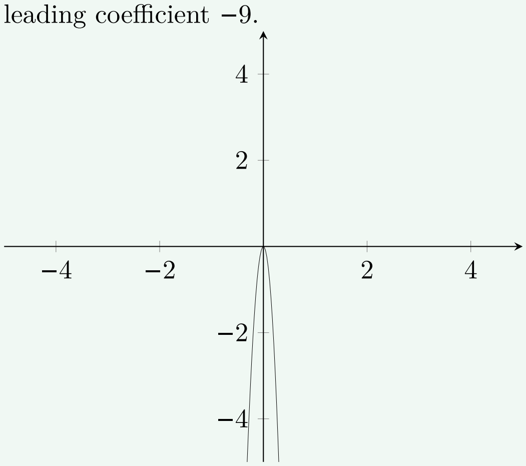
The leading term is $3x^3$, so the leading coefficient is 3 and degree is 3.



As $x \to \infty$, $f(x) \to \infty$
As $x \to -\infty$, $f(x) \to -\infty$.

Example

Given $f(x) = -9x^2(x+6)(x-1)$. **Find the degree and leading coefficient. Describe the behavior of $f(x)$ as $x \to \pm\infty$.**
-9 is outside all the parenthesis, so $-9x^4$ will be the leading term. It is 4th degree polynomial with leading coefficient -9.



As $x \to \infty$, $f(x) \to -\infty$
As $x \to -\infty$, $f(x) \to -\infty$.

Zeros of Polynomails: X-intercepts, Solutions or Roots

If P is a polynomial function, then c is called a zero of P if $P(c) = 0$. In other words, the zeros of P are the solutions or roots of the polynomial equation $P(x) = 0$. If P is a polynomial and $c$is a real number, then the following are equivalent:

1. c is a zero of P.
2. $x = c$ is a solution of the equation $P(x) = 0$.
3. $x - c$ is a factor of $P(x)$.
4. $x = c$ is an x-intercept of the graph of P.

A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing. The y-intercept is the point at which the function has an input value of zero. The x-intercepts are the points at which the output value is zero.

Turning Points and x-intercepts

The degree of a polynomial function helps us to determine the number of x-intercepts and the number of turning points. A polynomial function of $n-$th degree is the product of n factors, so it will have at most n roots or zeros, or x-intercepts. The graph of the polynomial function of degree n must have at most $n - 1$ turning points.

Multiple Zero – a zero of a linear factor that is repeated in the factored form of the polynomial.
Multiplicity of a Zero – the number of times the related linear factor is repeated in the factored form of a polynomial. The multiplicity of zero impacts the behavior of the graph around the x-intercept (bounce, cross)

Multiplicity of Zeros

In general, if c is a real zero of multiplicity k of a polynomial f $\big($ alternatively, if $(x - c)^k$ is a factor of f$\big)$, the graph of f will touch the x-axis at $(c, 0)$ and :

- Cross through x-axis if k is odd.
- stay on the same side of x-axis if k is even.

Further, if $k > 1$, the graph of f will "flatten out" near $(c, 0)$.

Example

Example 1: Consider the following function $f(x) = (x+3)(x-4)^5$.
a. Find the degree and y-intercept.
b. Find the x-intercept at which f crosses the x-axis.
c. Find the zeros at which f flattens out.

a. This is a 6th degree polynomial. To find the y-intercept we set, $x = 0$,;
$f(0) = (0+3)^2(0-4)^5 = -9216$.
y-intercept = $(0, -9216)$.

b. zeros with odd multiplicities, so $(-3, 0), (4, 0)$ crosses the x-axis.

c. f flattens out at $(4, 0)$

Example 2: Consider the following function $f(x) = (x+4)(x-2)^3$.
a. Find the degree and y-intercept.
b. Find the x-intercept at which f crosses the x-axis.
c. Find the zeros at which f flattens out.

a. This is a 4th degree polynomial. To find the y-intercept we set, $x = 0$,;
$f(0) = (0+4)(0-2)^3 = 4(-8) = -32$.
y-intercept = $(0, -32)$.

b. Both multiplicities are odd, so $(-4, 0), (2, 0)$ crosses the x-axis.

c. f flattens out at $(2, 0)$ because $3 > 1$.

3 Power Functions

A power function is a function that can be represented in the form

$$f(x) = kx^p$$

where k and p are real numbers, and k is known as the coefficient.

Examples of Power Functions

1. Constant function: $f(x) = 1$
2. Linear Function: $f(x) = x$
3. Quadratic Function : $f(x) = x^2$
4. Cubic Function: $f(x) = x^3$
5. Reciprocal Function: $f(x) = \frac{1}{x}$
6. Square root Function: $f(x) = \sqrt{x} = x^{\frac{1}{2}}$
7. Cube root Function : $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$
8. More examples:
   (a) $f(x) = \frac{1}{x^{4.2}}, f(x) = x^{-4}, f(x) = \sqrt[3]{x^2}, f(x) = \sqrt[4]{x}$
9. Examples that is not a power function
   (a) Function $f(x) = 2^x$. A power function contains a variable base raised to a fixed power. This function has a constant base raised to a variable power. This is called an exponential function, not a power function.
   (b) $f(x) = x^2 + 1, f(x) = 3^{x-17}, f(x) = x^3 + x^9$

4 Graphing Polynomial Functions

How to Graph a Polynomial from given equation

Given a polynomial function, sketch the graph.

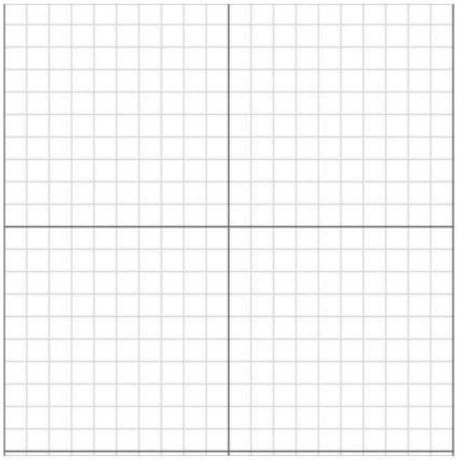
1. Find the intercepts.
2. Use the multiplicities of the zeros to determine the behavior of the polynomial at the x-intercepts. (Odd power: cross, Even power: bounce)
3. Determine the end behavior by examining the leading term.
4. Use the end behavior and the behavior at the intercepts to sketch a graph.
5. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
6. Find some test points

Example Graph $f(x) = y = -(x+2)(x-1)^2(x-3)^2$

Find x-intercepts:
$x =$ ___________, ___________, ___________

To find y-intercept, plug $x =$ ______ into ______
$f($______$) = -($______$+2)($______$-1)^2($______$-3)^2$

Leading coefficeint= ___________



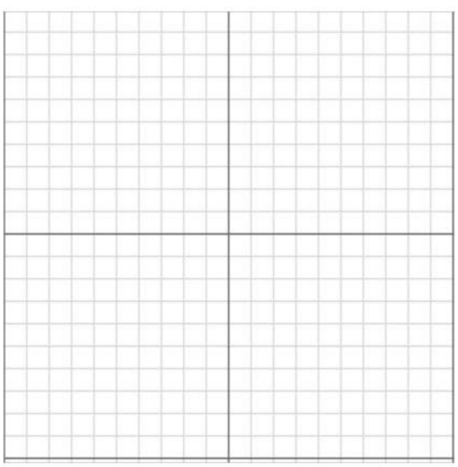
Key features
As $x \to \infty$, $f(x) \to$ ___________
As $x \to -\infty$, $f(x) \to$ ___________.
At $x =$ ______, $f(x) =$ ______, $x - axis$ (cross /bounce/flattens)
At $x =$ ______, $f(x) =$ ______, $x - axis$ (cross / bounce/falttens)
At $x =$ ______, $f(x) =$ ______, $x - axis$ (cross / bounce/faltterns)
Y-intercept ___________
X-intercepts ___________
Test points: ___________, ___________

Example Graph $f(x) = y = -4(x+2)(x+1)(x-1)$

Find x-intercepts:
$x =$ ___________, ___________, ___________

To find y-intercept, plug $x =$ ______ into ______
$f($______$) = -($______ $+ 2)($______ $+ 1)($______ $- 1)$

Leading coefficeint= ___________



Key features
As $x \to \infty$, $f(x) \to$ ___________
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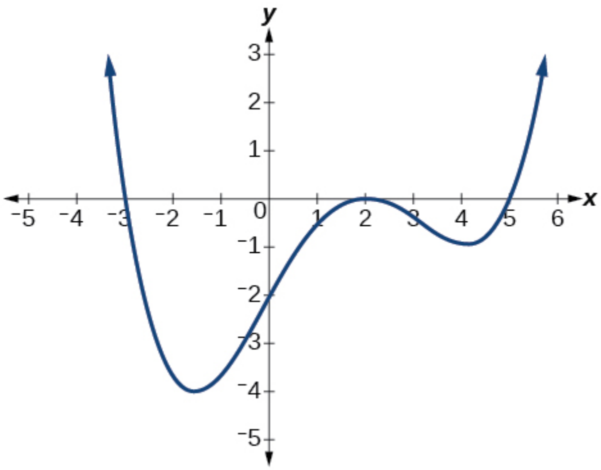
At $x =$ ______, $f(x) =$ ______, $x - axis$ (cross / bounce/faltterns)
Y-intercept ___________
X-intercepts ___________
Test points: ___________, ___________

Now we will do the opposite: Given the graph, find the equation of the polynomial
Write an expression in factored form of the polynomial of least possible degree graphed below:



Key features
End behavior:
As $x \to \infty$, $f(x) \to$ ______
As $x \to -\infty$, $f(x) \to$ ______.
Degree of Polynomial = ____________ (odd or even)
X-intercepts:
At x = ______, $f(x)$ = ______, $x - axis$ (cross /bounce/flattens), multiplicity= ______
At x = ______, $f(x)$ = ______, $x - axis$ (cross / bounce/falttens), multiplicity= ______
At x = ______, $f(x)$ = ______, $x - axis$ (cross / bounce/faltterns), multiplicity= ______
Write answer in factored form : $f(x)$ = $a(x$______)———(x______)———(x______)———
Y-intercept = ______
use Y-intercept (or test points) to find a: leading coefficient

Find the equation of the 3rd degree polynomial shown in Figure 1

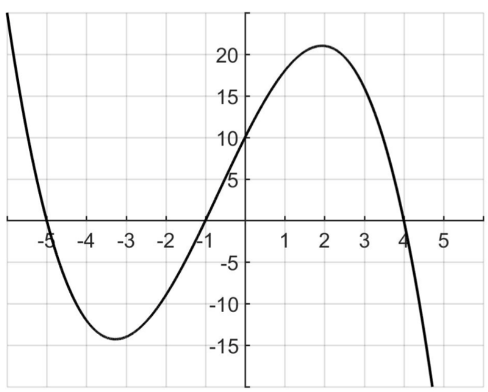


Figure 1: $y = f(x)$

Key features
End behavior:
As $x \to \infty$, $f(x) \to$ ______
As $x \to -\infty$, $f(x) \to$ ______.
Degree of Polynomial = ____________ (odd or even)
X-intercepts:
At x = ______, $f(x)$ = ______, $x - axis$ (cross /bounce/flattens), multiplicity= ______
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Write answer in factored form : $f(x)$ = $a(x$______)———(x______)———(x______)———
Y-intercept = ______
use Y-intercept (or test points) to find a: leading coefficient

Summary

1. Polynomial functions of degree 2 or more are smooth, continuous functions. They do not have holes or asymptotes.
2. To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero.
3. Another way to find the x-intercepts of a polynomial function is to graph the function and identify the points where the graph crosses the x-axis.
4. The multiplicity of a zero determines how the graph behaves at the x-intercept.
5. The graph of a polynomial will cross the x-axis at a zero with odd multiplicity.
6. The graph of a polynomial will touch and bounce off the x-axis at a zero with even multiplicity.
7. The end behavior of a polynomial function depends on the leading term.
8. The graph of a polynomial function changes direction at its turning points.
9. A polynomial function of degree n has at most $n-1$ turning points.
10. To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most $n-1$ turning points.
11. Graphing a polynomial function helps to estimate local and global extremas.