River Parishes Community College

Math 1100: College Algebra

Functions

3.1-3.2 Function, its Notation, Domain and Range

Semester
Fall/Spring Year

Department
Physical Science: Math

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Learning Objectives

In this section, you will learn:
♣ Determine Whether an equation Represents a Function.
♣ Find the Domain of a Function Defined by an Equation
♣ Graph the Functions Listed in the Library of Functions
♣ Graph Piecewise Functions

1 Function and its Notations

Quite often, a function is written as a equation, for example, $y = x^2 - 2x + 1$. The same function can be written as $f(x) = x^2 - 2x + 1$, which is called functional notation. Thus $f(x)$ is just another name for y. As a specific example, $f(3)$ means the y-value of the function when $x = 3$. This function can be found by substiuting each x-variable in the expression $x^2 - 2x + 1$ by 3. We write:

$$f(3) = 3^2 - 2(3) + 1 = 9 - 6 + 1 = 4$$

1.1 Domain and Range

In the above example, note that there is an order pair (x, y). When $x = 3$, the function yields $y = 4$. In other words for an input 3, the output is 4. We define all possible x-values as a **domain** of the the function and all possible y-values as a **range** of this function. For example, the domain of $f(x) = x$ is the entire real line i.e $(-\infty, \infty)$ in interval notation since $f(x)$ is defined for all values of x. IN FACT, DOMAIN OF ANY POLYNOMIAL FUNCTION IS ALL REAL NUMBERS.

However, not all functions enjoy entire real line as their domain. For example: $f(x) = \frac{1}{x}$ is not defined at $x = 0$. However, the function is defined for every other values of x. In other words, the domain of the function $f(x) = \frac{1}{x}$ is all real numbers except $x = 0$. In the interval notation, we write this domain as $(-\infty, 0) \cup (0, \infty)$. Here are two useful Principles:

TWO USEFUL PRINCIPLE FOR FINDING DOMAIN AND RANGE

For $\frac{1}{x}$ to be defined, x must be a non-zero quantity, i.e. $x \neq 0$.
For $\sqrt{x}$ to be defined as a real number, x must be a non-negative number i.e $x \geq 0$.

Example 1 Find the Domain and Range for $T = \{(3, -5), (1, -5), (1, -1), (1, -3)\}$
Solution:
Domain: all x values so$\{1, 3\}$
Range: all y-values , so $\{-5, -1, 1\}$

Example 2 Find the domain of $f(x) = \frac{x^2}{x^2 - x - 6}$. Express the exact answer in interval Notation.
Solution:
The numerator x^2 is always defined. The domain of $f(x)$ is all x-values for which denominator $x^2 - x - 6$ is nonzero. To make it easier, we will find all the x values for which the denominator is 0, so we solve:

$$x^2 - x - 6 = 0$$

$$(x - 3)(x + 2) = 0$$

$$x = 3, -2$$

The domain is all real numbers except 3 and -2., i.e. $\mathbb{R} - \{-2, 3\}$. In interval notation, we write $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.

Example 3 Find the domain of $f(x) = \frac{\sqrt{-2x+5}}{x+6}$. Express the exact answer in the interval notation.
Solution:
The denominator and numerator both are not defined for all real number. The numerator is defined for all the x for which $-2x + 5 \geq 0$. Likewise the we need denominator $x + 6$ to be nonzero for the function to be defined.

$$-2x + 5 \geq 0$$

$$-2x \geq -5$$

Dividing both sides by -2 flips the inequality as well

$$x \leq \frac{5}{2}$$

This means the numerator is defined for all values of x less than or equal to $\frac{5}{2}$. Also note that the denominator is zero when $x = -6$. This means our function can not be defined when $x = -6$. So we have to take out -6 from the previous list i.e. Domain is $\{x : x \leq \frac{5}{2}\} - \{-6\}$. In interval notation, $(-\infty, -6) \cup (-6, \frac{5}{2}]$.

1.2 Characteristic of a Function

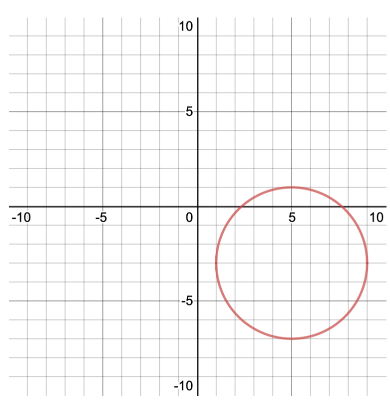
For any input x, the function yields exactly one output y. For an equation to be a function, one input can not have two or more outputs.

Example 4 Find the Domain and Range for $T = \{(3, -5), (1, -1), (1, -3)\}$ Is this a function?
Solution:
No this is not a function because input $x = 1$ has two outputs $y = -1, -3$.

1.2.1 Vertical Line Test

Any two variable equation can be graphed easily by selecting some points that satisfy the equations. However, all graphs do not represent the function. There is any easy test to tell whether a given graph is a function or not, and the test is called **Vertical line test**. That is if you can draw a vertical line anywhere in the graph and it touches two points on that graph, then it is not a function because it fails to preserve the characteristic of a function: One input can not have two or more outputs.

Example 5 Find the domain and range. Is this a function?



Solution:
Domain = $[1, 9]$
Range = $[-7, 1]$.
No this is not a function

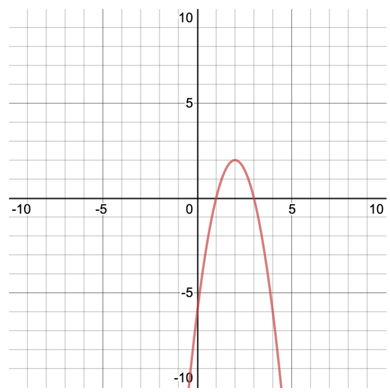
1.3 One-to-One function

They are pretty nice family of functions. In this family, One input has only one output and one output has only one input. **That is no two different x values has the same y values.**. We will see this again when we study Inverse functions because a function has an inverse if it is one-ot-one.

1.3.1 Horizontal line Test

Given a graph of two variable, we can easily test for one-to-one function. If one can draw a horizontal line passing through two or more points in a graph, then it is not a one-to-one function.

Example 6 Find the domain and range. Is this a function? Is this an one-to-one function?



Solution:
Domain = $(-\infty, \infty)$
Range = $(-\infty, 2]$. This is a function. However, it fails a horizontal line test. Hence this is not an example of one-to-one function.

2 Piecewise Functions

A **Piecewise Function** is a function defined by two or more equations. Each piece of the function applies to a different part of its domain. The simplest example is the absolute value function .

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

The expression $-x$ represents the y values of this function for all negative numbers only $x < 0$.
The expression x represents the y value of this function for all positive numbers and zero only . $x \geq 0$.

Example 7 Consider the following piecewise-defined function.

$$F(x) = \begin{cases} 2x + 3 & if\ x \leq -1 \\ x^2 - 2x + 3 & if\ x > -1 \end{cases}$$

1: Evaluate this function at $x = -1$.

2: Evaluate this function at $x = -4$.

3 : Evaluate this function at $x = 1$.
Solution:
1: Observe that $F(x) = 2x + 3$ when $x = -1.$, So we $F(-1) = 2(-1) + 3 = -2 + 3 = 1$.

2: Since $-4 < -1$, We use $F(x) = 2x + 3$, So $F(-4) = 2(-4) + 3 = -8 + 3 = -5$.

3: Since $1 > -1$, We use $F(x) = x^2 + 2x + 3$, So $F(1) = 1^2 - 2 * 1 + 3 = 1 - 2 + 3 = 2$.

Example 8 Consider the following function

$$f(x) = \begin{cases} 2x - 2 & if\ x \leq -1 \\ 2x^2 - 2x & if\ -1 \leq x < 6 \\ -\frac{2}{3}x + 7 & if\ x \geq 6. \end{cases}$$

Find $f(1), f(6)$ & $f(-3)$.
Solution:
1 lies in between $-1 \leq x < 6$ so, $f(1) = 2x^2 - 2x = 2.1^2 - 2.1 = 2 - 2 = 0$.

$6 \geq 6$, so $f(6) = -\frac{2}{3}x + 7 = -\frac{2}{3}6 + 7 = -4 + 7 = 3$.

$-3 \leq -1$, so $f(-3) = 2x - 2 = 2(-3) - 2 = -8$.

Example 9 Consider the following Function:

$$f(x) = \begin{cases} 2x - 3 & if\ x \leq -1 \\ -3x + 4 & if\ x > -1 \end{cases}$$

Find $f(-1), f(1)$ & $f(-5)$.

Solution:
$-1 \leq -1$, so $f(-1) = 2x - 3 = 2(-1) - 3 = -5$.

$1 > -1$, so $f(1) = -3x + 4 = -3 * 1 + 4 = 1$.

$-5 \leq -1$, so $f(-5) = 2x - 3 = 2 * (-5) - 3 = -13$.

Example 10 A museum charges \$6 per person for a guided tour with a group of 1 to 13 people or a fixed \$84 fee for a group of 14 or more people. Write a function relating the number of people, n, to the cost, C.
Solution:
There is only one boader line case in this problem i.e group of 13 or less and group of more than 13. For a group of 1 to 13, it is \$6 per person . Let n represent the number of person in this group, then the cost is $6n$. That is $C(n) = 6n, 1 \leq n \leq 13$.
For a group of 14 or more, it is flat \$84. That is $C(n) = 84, n \geq 14$.

$$C(n) = \begin{cases} 6n & 0 < x < 14 \\ 84 & x \geq 14 \end{cases}$$

Example 11 Given $f(x) = 2x^2 + 3x + 1$, find the exact value of $f(-3)$
Solution:

$$f(-3) = 2(-3)^2 + 3(-3) + 1 = 2(9) - 9 + 1 = 10$$

Example 12 Given, $f(x) = 3x^2 - 4x + 1$, find x if $f(x) = 21$.
Solution:
DO NOT PLUG IN $x = 21$ in the given equation, as that would be a major mistake. Set $f(x) = 21$ and solve as follows:

$$3x^2 - 4x + 1 = 21$$
$$3x^2 - 4x - 20 = 0$$

The above is a quadratic equation. BE VERY FAMILIAR with the methods of solving quadratic equations and other equations. For example, one can use the factoring method or the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. Lets use the factoring method :

$$(3x - 10)(x + 2) = 0$$
$$3x - 10 = 0, \ or \ x + 2 = 0$$
$$x = \frac{10}{3} \ or \ x = -2$$

Thus there are two values for which $f(x) = 21$.

Example 13 IMPORTANT EXAMPLE : Given $f(x) = 3x^2 - 4x + 1$, find $f(x + h)$. Expand and simplify your answer. Solution:
This problem illustrates one of the subtle point of functional notation. DO NOT simply add h to the right hand side of the equation to obtian $3x^2 - 4x + 1 + h$ as the answer for $f(x + h)$. In fact, $3x^2 - 4x + 1 + h$ is equal to $f(x) + h$, which is different form $f(x + h)$.
In order to find $f(x + h)$, replace all the x-terms on the right hand side of the given equation by $x + h$:

$$f(x + h) = 3(x + h)^2 - 4(x + h) + 1$$

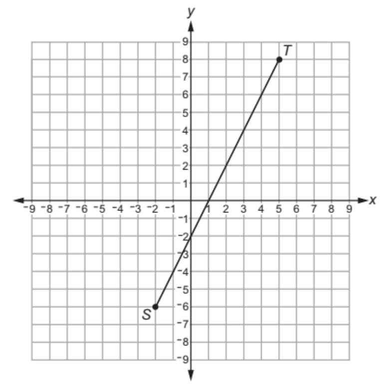
Now use rules of algebra such as the FOIL METHOD or special formulas to expand the right-hand side:

$$f(x + h) = 3(x^2 + 2xh + h^2) - 4x - 4h + 1$$
$$= 3x^2 + 6xh + 3h^2 - 4x - 4h + 1$$
$$f(x + h) = 3x^2 + 6xh + 3h^2 - 4x - 4h + 1$$

Example 14 Find the Domain for $f(x) = \sqrt{x - 4} + 9$. Write in Interval Notation
Solution:
Note that the domain of a square root function is the set of all positive real numbers, so,
$x - 4 \geq 0$
$x \geq 4$.
In Interval domain = $[4, \infty)$.

Example 15 Find the Domain for $f(x) = -\frac{\sqrt{x+2}}{14}$ Write in Interval Notation
Solution:
Note that the domain of a square root function is the set of all positive real numbers, so,
$x + 2 \geq 0$
$x \geq -2$.
In Interval domain = $[-2, \infty)$.

Example 16 Determine the implied Domain of the following function $f(x) = -\frac{10}{x+3}$.
Solution:
This is an example of rational function which we will study later. Remember that we don't want zero in the denominator. That means
$x + 3 \neq 0$.
$x \neq -3$.
Domain is anything except -3, so we write it interval notation as $(-\infty, -3) \cup (-3, \infty)$.

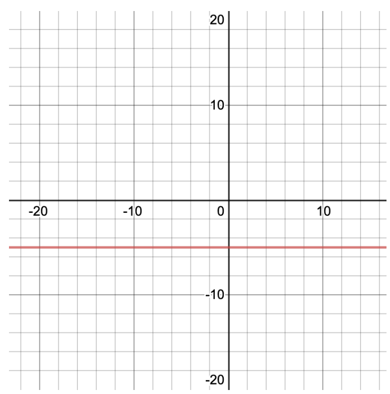


Example 17 Consider the following relation $2x - y = 2$

Find the domain and range. Is this a function?

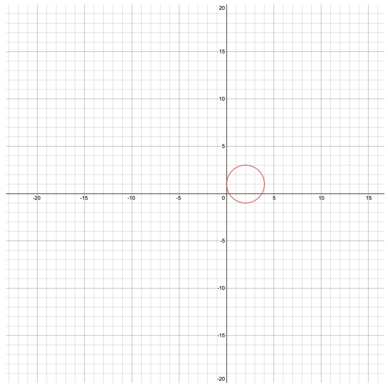
Solution:
Domain (all-x values) = $(-\infty, \infty)$.
Range (y-values)=) = $(-\infty, \infty)$.
Yes this is a function.

Example 18 Consider the following relation $y = -4$



Find the domain and range.
Solution:
Domain (all-x values) = $(-\infty, \infty)$.
Range (y-values)=) = 4
Yes this is a function.

Example 19 Find the domain and range.



Is this a function?
Solution Domain = $[0, 4]$
Range = $[-1, 3]$.
No this is not a function

Example 20 If $g(x) = -5x^2 + 3x$, find $g(x+1), g(x^2)$.
Solution:
$g(x+1)$ means replace all x with $x+1$ in $g(x) = 5x^2 + 3x$.
That is;
$g(x+1) = 5(x+1)^2 + 3(x+1)$.
$g(x+1) = 5(x+1)(x+1) + 3x + 3$.
$g(x+1) = 5(x^2 + 2x + 1) + 3x + 3$.
$g(x = 1) = 5x^2 + 10x + 5 + 3x + 3$.
$g(x+1) = 5x^2 + 13x + 8$.

Similarly, $g(x^2) = 5(x^2)^2 + 3(x^2)$.
$g(x^2) = 5x^4 + 3x^2$.

Example 21 Let $f(x) = \sqrt{x-4} + 7$. Find $f(x-1)$.
Solution:
$f(x-1) = \sqrt{x-1-4} + 7$.
$f(x-1) = \sqrt{x-5} + 7$.

Example 22 Rewrite the relation as a function of x. $5x^2 + 2y = -3x - 2y$. Evaluate the function at $x = -2$.
Solution:
Rewriting as a function of x means isolate y.
Then, $5x^2 - 5x^2 + 2y + 2y = -3x - 2y + 2y - 5x^2$
$4y = -3x - 5x^2$
$y = -\frac{3}{4}x - \frac{5}{4}x^2$

Now $f(-2) = -\frac{3}{4}(-2) - \frac{5}{4}(-2)^2$.

$= \frac{6}{4} - \frac{5}{4} * 4$.

$= \frac{6}{4} - \frac{20}{4}$

$$= \frac{6-20}{4}.$$

$$= -\frac{-14}{4}.$$

$$= -\frac{7}{2}.$$